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GRAPH THEORETIC PARAMETERS ASSOCIATED WITH PBIB DESIGN VIA PARTIAL GEOMETRIES OF GENERALIZED POLYGON

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Abstract: Due to Feit and Higman [12], the (thick) generalized n-gons exist only for $n \in \{2, 3, 4, 6, 8\}$ and are apparently quite rare for $n = 6$ or 8. By virtue of the above fact, in this article, we investigate the generalized polygons which are strongly regular graphs and pseudo geometric graphs. Also, we obtain the parameters of partial geometry and partially balanced incomplete block (PBIB) designs with association scheme arising from classical graph theoretic parameters (covering, independence, domination and neighborhood number) on generalized polygons.

Keywords and Phrases: Graph, partial geometry, generalized polygons, pseudogeometric graph, partially balanced incomplete block design.

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1. Introduction

Let a graph $G = (V, E)$ be finite, simple, undirected, without loops and multiple lines. In general, we use $\langle S \rangle$ to denote the sub graph induced by the set of points S. Also, by recalling some classical parameters in graph theory as follows:

A set $S \subseteq V(G)$ is called a point covering set, if every line of G is incident to at least one point in S / independent set, if no two points in S are adjacent / dominating set, if every in point $V - S$ is adjacent to at least one point in S / neighborhood set, if $G = \cup_{v \in S}\langle N[v]\rangle$ where $\langle N[v]\rangle$ is the subgraph of G induced by v and all points adjacent to v. The smallest cardinality among all point covering / dominating / neighborhood set is called a point covering number $\alpha(G)$ / domination number $\gamma(G)$ / neighborhood number $\eta(G)$, and the largest cardinality of independent set S is called a point independence number $\beta(G)$ of a graph G. A minimum covering / dominating / neighborhood set S of a graph G with $|S| = \alpha(G)/\gamma(G)/\eta(G)$ is called a $\alpha/\gamma/\eta$ - set and a maximum independent set S of a graph G with $|S| = \beta(G)$ is called a β- set of G. For graph-theoretical terminology and notation not defined here we follow [13]. Also, for more details on above parameters, we refer to [6], [14] and [21].

For an incidence structure or incidence system in geometry, we consider objects as points (vertices) and lines (edges), with point-line incidence structures (P, L, I) or usually (P, L) in which the set of points P, the set of lines L and incidence relation as $I \subseteq P \times L$ called incidence structure or incidence relation. This traditional affine and projective geometry are highly influenced to applied statistics and combinatorics. For more details, we refer to [22].

A partial geometry $pg(s+1, t+1, \varrho)$ is a point-line incidence structure having every line has $s+1$ points, every point is incident with exactly $t+1$ lines, and for every line L and every point p not on line L there are exactly α lines through p that meet L. The notation defined by us follows [1]. It seems to be more natural than others since $s+1$, $t+1$ and ϱ correspond to the objects counted in the above properties.

The point graph of a partial geometry has the points of geometry and two points form an line if and only if they are on a common line. The point graph of a partial geometry $pg(s+1, t+1, \varrho)$ is strongly regular graph $srg(p, l, \sigma, \mu)$ where $p = \frac{(s+1)(st+\varrho)}{\varrho}$, $l = s(t+1)$, $\sigma = s - 1 + \varrho(t+1)$ and $\mu = \varrho(t+1)$.

If a strongly regular graph has parameters such that it could be the point graph of a partial geometry we call the strongly regular graph pseudo-geometric and if it is indeed the point graph of a partial geometry we call the strongly regular graph geometric. A pseudo-geometric strongly regular graph is not necessarily geometric.

Theorem 1.1. [4] *If a strongly regular graph is pseudo-geometric corresponding to* $(s+1, t+1, \alpha)$ *and* $2(s+1) > t(t+1) + \alpha(t+2)(t^2+1)$ *then the graph is geometric.*

In 1963, Bose [1] introduced the concept of a partial geometry in order to study large cliques in strongly regular graphs. Several strong necessary conditions for the existence of strongly regular graphs also have consequences for the existence question for partial geometries.

In 1974, Jacques Tits [17] introduced the partial geometry for the classification of finite groups. In his fundamental work, he introduced point-line incidence structure (P, L), which is a generalized polygon or generalized n-gon (where $n \geq 2$) if its incidence graph Γ satisfies the following properties:

(i) A set of points of Γ is partitioned into points and lines, every line of Γ joins a point with a line, there fore it is a bipartite.

(ii) Γ has girth n and diameter $2n$.

(iii) For every line has $s + 1 \geq 2$ points, every point is on $t + 1 \geq 2$ lines.

The order of the generalized polygon is the pair (s, t) or simply we can denoted by s if $s = t$. The following well known result is the main inspiration to further development on this article.

Theorem 1.2. (Feit and Higman, [12]) *Finite thick generalized n-gons exist only for $n \in \{2, 3, 4, 6, 8\}$.*

For more details on graph-theoretic and geometric terminologies, we refer to [5], [10], [11], [12] and [18].

In 1939, the binary class, equireplicate class and proper designs class, which are called the partially balanced incomplete block (PBIB) designs, introduced by Bose and Nair [2]. On the basis of their association schemes, they can be grouped into different types via higher association schemes of PBIB designs. As per the type of underlying association scheme, every individual groups has further subgroups.

An association scheme with m classes satisfies the following conditions for given ν points (vertices or elements or objects):

(i) If the associates are symmetric, then any two points are m^{th} associates, where $1 \leq k \leq m$.

(ii) Each point x contains n_k k^{th} associates, the number n_k being independent of x.

(iii) If two points x and y are k^{th} associates, then the number of points which are both a^{th} associates of x and b^{th} associates of y is p^k_{ab} and is independent of the k^{th} associates x and y. Hence $p^k_{ab} = p^k_{ba}$.

Based on the above association scheme on ν points, a PBIB design is defined as follows.

A PBIB design is an arrangement of ν points into ρ sets (called blocks) of size g where $g < \nu$ such that

(i) Every point is contained in exactly r blocks.

(ii) Each block contains g distinct points.

(iii) Any two points which are k^{th} associates occur together in exactly λ_k blocks, where $1 \leq k \leq m$.

The numbers ν, ρ, r, g, λ_1, λ_2, ..., λ_m are the parameters of the first kind, whereas the numbers n_1, n_2, ..., n_m, p^k_{ab} (a, b, $k = 1, 2, \ldots, m$) are the parameters of the second kind; for more details, we refer [1], [2], [7] and [8].

The above PBIB designs has many applications to cluster sampling and are widely used in cryptology, digital fingerprint codes, many experimental situations such as the removal of lichens, weathering stability under ultraviolet radiations, an in-ground natural durability field test and one can refer to [16] and [19].

In this article, we investigate partial geometry which is also admitting pseudo-geometry, that is point graph of partial geometry. Here pseudo geometry and it's characteristics are verified for generalized digon of order s, generalized triangle, generalized quadrangle of order (s,t), generalized hexagon of order (s,t) and generalized octagon of order (s,t). For these, we apply PBIB designs with association schemes using graph theoretic parameters.

2. Generalized digon of order s

A generalized digon of order s is an incidence structure with $s+1$ points and the same number of blocks. Thus every point is incident with every block. Therefore, this incidence graph is known as regular complete bipartite graph $K_{s+1,s+1}$.

Theorem 2.1. *Let G be a strongly regular graph of generalized digon of order (s,t). Then G is the point graph of a partial geometry if and only if $s = t = 1$.*

Proof. Let G be a generalized digon of order (s,t). We have the following cases arises,

Case 1. If $s = 1$ and $t \neq 1$, then the graph G is not a generalized digon of order (s,t), which is a contradiction to our assumption.

Case 2. If $s \neq 1$ and $t = 1$, then it is also similar to the case 1.

Case 3. If $s = 1$ and $t = 1$, then it satisfies all the parameters of strongly regular graph. Further, it follows the definition of generalized digon of order (s,t) and partial geometry $pg(s+1, t+1, \varrho)$ along with the parameters of point graph of a partial geometry. Therefore $\varrho = 1$.

Case 4. If $s \neq 1 (s \geq 2)$ and $t \neq 1 (t \geq 2)$, then it does not satisfies the parameters of strongly regular graphs. By the definition of the generalized digon of order (s,t)

and partial geometry $pg(s+1, t+1, \varrho)$ along with the parameters of point graph of a partial geometry. Therefore the values of ϱ is not consistent.

Thus the result follows.

Theorem 2.2. *The collection of all lines, α, β, γ and η-sets of a generalized digon of order s forms the parameters of partial geometry and PBIB design with 2-association scheme are given in Table 1.*

Collections	s	t	ν	ρ	g	r	λ_1	λ_2
Lines	≥ 1	≥ 1	$2(s+1)$	$s(s+1)^2$	$s+1$	$(s+1)(t+1)$	$3s-1$	$3s-2$
α	≥ 1	≥ 1	$2(s+1)$	2	$s+1$	1	1	0
β	≥ 1	≥ 1	$2(s+1)$	2	$s+1$	1	1	0
γ	≥ 2	≥ 2	$2(s+1)$	$(t+1)^2$	2	$s+1$	0	1
η	≥ 2	≥ 2	$2(s+1)$	2	$s+1$	1	1	0

Table 1. The parameters of partial geometry and PBIB design of generalized digon of order s.

Proof. Let G be a graph of generalized digon of order s with the points are labeled as $v_1, v_2, v_3, \ldots, v_n$ in random order. Then, the two distinct points v_i and v_j are said to be first associated if they are nonadjacent and second associated, otherwise. By using this association scheme, we have the parameters of second kind are

$$P^k = \begin{pmatrix} p_{11}^1 & p_{12}^1 \\ p_{21}^1 & p_{22}^1 \end{pmatrix} = \begin{pmatrix} 0 & s \\ s & 0 \end{pmatrix} = \begin{pmatrix} 0 & t \\ t & 0 \end{pmatrix}$$

and

$$P^k = \begin{pmatrix} p_{11}^2 & p_{12}^2 \\ p_{21}^2 & p_{22}^2 \end{pmatrix} = \begin{pmatrix} 1+s & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 1+t & 0 \\ 0 & 0 \end{pmatrix}.$$

Also, the parameters of first kind are

(i) The collection all lines are given by $\nu = 2(s+1)$ or $2(t+1)$, $\rho = s(s+1)^2$, $g = s+1$ or $t+1$, $r = (s+1)(t+1)$, $\lambda_1 = 3s-1$ and $\lambda_2 = 3s-2$(where $s, t \geq 1$).

(ii) The collection all α - sets are given by $\nu = 2(s+1)$, $\rho = 2$, $g = s+1$, $r = 1$, $\lambda_1 = 1$ and $\lambda_2 = 0$ (where $s, t \geq 1$).

(iii) The collection all β - sets are given by $\nu = 2(s+1)$, $\rho = 2$, $g = s+1$, $r = 1$, $\lambda_1 = 1$ and $\lambda_2 = 0$ (where $s, t \geq 1$).

(iv) The collection all γ - sets are given by $\nu = 2(s+1)$, $\rho = (t+1)^2$, $g = 2$, $r = s+1$, $\lambda_1 = 0$ and $\lambda_2 = 1$ (where $s, t \geq 2$).

(v) The collection all η - sets are given by $\nu = 2(s+1)$, $\rho = 2$, $g = s+1$, $r = 1$, $\lambda_1 = 1$ and $\lambda_2 = 0$ (where $s, t \geq 2$).

Now, we obtain the computed values of some classical graph theoretic parameters of generalized digon of order s.

Theorem 2.3. *Let G be a graph of generalized digon of order s. Then*

(i) $\alpha(G) = \beta(G) = \eta(G) = s+1$,

(ii) $\gamma(G) \leq s+1$.

Proof. (i) By the definition of generalized digon of order s, the desired result follows.
(ii) If G is a graph of generalized digon of order of s, then $G \cong K_{s+1,+1}$. This implies that, for any two adjacent points dominates all the points. Hence $\gamma(G) \leq s+1$ and the equality holds for $s = 1$.

3. Generalized Triangle

A projective plane is a geometry, which satisfies any two lines intersect in exactly one point. Since a projective plane is a special case of generalized polygon or generalized n-gon when $n = 3$ and these are equivalent to generalized triangle. If $s = t = 1$, then it is called a thin projective plane. Otherwise, it is called a thick projective plane.

Theorem 3.1. *A thin projective plane is not a strongly regular graph. Therefore it is not a pseudo-geometric graph.*
Proof. Assume $n = 3$ in a generalized polygon, it becomes a generalized triangle or projective plane. If the order of the projective plane is 1, then it becomes a finite projective plane. It is a strongly regular graph and also a partial geomatry. But it does not satisfies the parameters of the point graph of a partial geometry.

Theorem 3.2. *The collection of all lines, α, β, γ and η-sets of a thin generalized triangle of order (s, t) forms the parameters of partial geometry and PBIB design with 1-association scheme are given in Table 2.*

Collections	s	t	ν	ρ	g	r	λ_1
Lines	1	1	$s(s+1)+1$	$t(t+1)+1$	$s+1$	$s+1$	1
α	1	1	$s(s+1)+1$	$t(t+1)+1$	$s+1$	$s+1$	1
β	1	1	$s(s+1)+1$	$t(t+1)+1$	1	1	0
γ	1	1	$s(s+1)+1$	$t(t+1)+1$	1	1	0
η	1	1	$s(s+1)+1$	$t(t+1)+1$	$s+1$	$s+1$	1

Table 2. The parameters of partial geometry and PBIB design of a thin generalized triangle of order (s,t).

Proof. Let G be a thin generalized triangle of order (s,t) with the points labeled as $v_1, v_2, \ldots, v_n$ in random order. Then the two distinct points v_i and v_j are said to be first associated if they are nonadjacent and second associated, otherwise. By using this association scheme, we have the parameters of second kind are

$$P^k = \begin{pmatrix} p_{11}^1 & p_{12}^1 \\ p_{21}^1 & p_{22}^1 \end{pmatrix} = \begin{pmatrix} 0 & s \\ s & 0 \end{pmatrix} = \begin{pmatrix} 0 & t \\ t & 0 \end{pmatrix}$$

and

$$P^k = \begin{pmatrix} p_{11}^2 & p_{12}^2 \\ p_{21}^2 & p_{22}^2 \end{pmatrix} = \begin{pmatrix} 1+s & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 1+t & 0 \\ 0 & 0 \end{pmatrix}.$$

Also, the parameters of first kind are

(i) The collection of all lines are given by $\nu = s(s+1)+1$, $\rho = t(t+1)+1$, $g = s+1$, $r = s+1$ and $\lambda_1 = 1$ (where $s,t = 1$).

(ii) The collection of all α - sets are given by $\nu = s(s+1)+1$, $\rho = t(t+1)+1$, $g = s+1$, $r = s+1$ and $\lambda_1 = 1$ (where $s,t = 1$).

(iii) The collection of all β - sets are given by $\nu = s(s+1)+1$, $\rho = t(t+1)+1$, $g = 1$, $r = 1$ and $\lambda_1 = 0$ (where $s,t = 1$).

(iv) the collection of all γ - sets are given by $\nu = s(s+1)+1$, $\rho = t(t+1)+1$, $g = 1$, $r = 1$ and $\lambda_1 = 0$ (where $s,t = 1$).

(v) The collection of all η- sets are given by $\nu = s(s+1)+1$, $\rho = t(t+1)+1$, $g = s+1$, $r = s+1$ and $\lambda_1 = 1$ (where $s,t = 1$).

Remark 3.3. *For the generalized digon of order s and for the thin projective plane, the parameters of second kind are same due to $s = t$.*

Theorem 3.4. *The projective planes are not a strongly regular graphs and also not the point graph of a partial geometry.*
Proof. Since the projective plane are not strongly regular graphs except for order 1. By Theorem 3.1, the desired result follows.

Remark 3.5. *For a thick projective plane of order (s, t), the parameters of PBIB design and its association scheme does not exists.*

4. Generalized Quadrangle

A generalized quadrangle is a point-line incidence structure satisfying the following conditions:

(i) For some $s \geq 1$, every line has $s + 1$ points.

(ii) For some $t \geq 1$, every point lies on $t + 1$ lines.

(iii) If p is a point not on a line L, then there is a unique line through p meeting L.

A generalized quadrangle of order s if $s = t$ or simply generalized quadrangle of order (s, t).

If $s = 1$ or $t = 1$ are called the thin generalized quadrangle of order (s, t). A generalized quadrangle of order $(s, 1)$, a grid with $(s+1)^2$ points and $2(s+1)$ lines, that is, a 2-net of order $s+1$. A generalized quadrangle of order $(1, t)$, a dual grid with two sets $(t + 1)$ points and $(t + 1)^2$ lines joining every point in one set with every pair in the other set. For more details, we refer to [3], [9], [20] and [23].

Theorem 4.1. *If a generalized quadrangle of order $(1, t)$ is a strongly regular graph then it is the pseudo-geometric graph.*
Proof. Let G be a generalized quadrangle of order (s, t). If $s = 1$, then it becomes a thin generalized quadrangle of order $(1, t)$. But, according to the partial geometry $pg(s+1, t+1, \varrho)$ with $\varrho = 1$ is called the generalized quadrangle. By the definition of thin generalized quadrangle of order $(1, t)$, the graph G is a dual grid or complete bipartite graph, which is a strongly regular graph. Due to the Brouwer and van Lint [4], we have the parameters of a strongly regular graph is same as the parameter of the point graph of a partial geometry. Hence the thin generalized quadrangle of order $(1, t)$ is the point graph of a partial geometry.

Theorem 4.2. *For every generalized quadrangle of order $(1, t)$ is a generalized digon of order s.*
Proof. Since the parameters of partial geometry $pg(s + 1, t + 1, \varrho)$ with $\varrho = 1$ is a generalized quadrangle. Consider the generalized quadrangle of order $(1, t)$ consists

of two sets of $t+1$ points and $(t+1)^2$ lines, which is a dual grid or a complete bipartite graph. Similarly, a generalized digon of order s, consists of $s+1$ points and same number of blocks and every point is incident with every block, so it is also a complete bipartite graph. Therefore a generalized quadrangle of order $(1,t)$ is a generalized digon of order s.

Remark 4.3. *Converse of the above theorem is not true.*

For instance, suppose a generalized digon of order s is a generalized quadrangle of order $(1,t)$. Since the generalized digon only for $s=1$ and $t=1$, the values of ϱ varies from one another. Hence, it is not the point graph of a partial geometry. Further, in the generalized quadrangle of order $(1,t)$, the value of ϱ is one. Thus, the parameters of strongly regular graph of the generalized quadrangle of order $(1,t)$ is same as the parameters of the point graph of a partial geometry, which is a contradiction to our assumptions.

Therefore a generalized digon of of order s need not be a generalized quadrangle of order $(1,t)$.

Here, we obtain the computed values of some classical graph theoretic parameters of a generalized quadrangle of order (s,t).

Theorem 4.4. *Let G be a graph of generalized quadrangle of order $(1,t)$. Then*

(i) $\alpha(G)=\beta(G)\geq s+1$,

(ii) $\gamma(G)=s+1$,

(iii) $\eta(G)\geq s+1$.

Proof. (i) Since a generalized quadrangle of order $(1,t)$ is isomorphic to a complete bipartite graph, which implies all non adjacent points covers all the lines. Similarly, for point independence also. Hence $\alpha(G)=\beta(G)\geq s+1$ and the equality holds for $s=1$ and $t=1$.

(ii) By the definition of generalized quadrangle of order (s,t) with $s=1$ and $t\geq 1$, the desired result follows.

(iii) This is similar to case (i).

Theorem 4.5. *A generalized quadrangle of order $(1,t)$ is a strongly regular graph but a generalized quadrangle of order $(s,1)$ is not a strongly regular graph.*

Proof. Let G be a generalized quadrangle of order $(1,t)$. By Theorem 4.1, we have the generalized quadrangle of order (s,t) with $s=1$, then it becomes a thin generalized quadrangle of order $(1,t)$. This implies that the parameters of a strongly regular graph varies from one another. Therefore a generalized quadrangle of order $(s,1)$ is not a strongly regular graph.

Remark 4.6. *A thin generalized quadrangle of order* $(s, 1)$ *is pseudo-geometric graph. Therefore, the parameters of PBIB design and its association scheme does not exists.*

Theorem 4.7. *In a generalized quadrangle of order* (s, t)*, the partial geometries need not be a pseudo-geometric graph.*
Proof. Let $pg(s+1, t+1, \varrho)$ be a partial geometry of generalized quadrangle with $\varrho = 1$. If $s = 1$ or $t = 1$ or both, then it is a thin generalized quadrangle of order $(s, 1)$ or $(1, t)$ or $(1, 1)$; otherwise, it is thick. Further, a thin generalized quadrangle of order $(1, t)$ satisfies all the parameters of strongly regular graph. Clearly, it is a point graph of a partial geometry. Consequently, in a generalized quadrangle of order (s, t) with $s \geq 1$ and $t \geq 1$, the parametric values of strongly regular graphs are varying. Therefore in a generalized quadrangle of order $(s, 1)$, a partial geometry need not be a pseudo-geometric graph.

Remark 4.8. *A generalized quadrangle of order* $(1, t)$ *is a generalized digon of order* s*. But a generalized digon of order* s *need not be generalized quadrangle of order* $(1, t)$

Theorem 4.9. *The collection of all lines,* α*,* β*,* γ *and* η*-sets of a generalized quadrangle of order* $(1, t)$ *forms the parameters of partial geometry and PBIB design with* 2*-association schemes are given in Table 3,*

Collections	s	t	ν	ρ	g	r	λ_1	λ_2
Lines	≥ 1	≥ 1	$2(s+t)$	$2(s+t)^2$	$s+1$	$2(s+t)$	$s-1$	$s+1$
α	1	≥ 1	$2(t+1)$	2	$t+1$	1	1	0
β	1	≥ 1	$2(s+1)$	2	$s+1$	1	1	0
γ	1	≥ 2	$2(t+1)$	$(t+1)^2$	2	$t+1$	0	1
η	1	≥ 2	$2(t+1)$	2	$t+1$	1	1	0

Table 3. The parameters of partial geometry and PBIB design of a generalized quadrangle of order (s, t).

Proof. Let G be a generalized quadrangle of order $(1, t)$ with points are labeled as $v_1, v_2, v_3, \ldots, v_n$ in random order. Then the two distinct points v_i and v_j are said to be first associated if they are nonadjacent and second associated, otherwise. By using this association scheme, we have the parameters of second kind are

$$P^k = \begin{pmatrix} p_{11}^1 & p_{12}^1 \\ p_{21}^1 & p_{22}^1 \end{pmatrix} = \begin{pmatrix} 0 & t \\ t & 0 \end{pmatrix}$$

and

$$P^k = \begin{pmatrix} p^2_{11} & p^2_{12} \\ p^2_{21} & p^2_{22} \end{pmatrix} = \begin{pmatrix} 1+t & 0 \\ 0 & 0 \end{pmatrix}.$$

Also, the parameters of first kind are as follows.

(i) The collection of all lines are given by $\nu = 2(s+t)$, $\rho = 2(s+t)^2$, $g = s+1$, $r = 2(s+t)$, $\lambda_1 = s-1$ and $\lambda_2 = s+1$ (where $s, t \geq 1$).

(ii) The collection of all α - sets are given by $\nu = 2(t+1)$, $\rho = 2$, $g = t+1$, $r = 1$, $\lambda_1 = 1$ and $\lambda_2 = 0$ (where $s = 1$, $t \geq 1$).

(iii) The collection of all β - sets are given by $\nu = 2(t+1)$, $\rho = 2$, $g = t+1$, $r = 1$, $\lambda_1 = 1$ and $\lambda_2 = 0$ (where $s = 1$, $t \geq 1$).

(iv) The collection of all γ - sets are given by $\nu = 2(t+1)$, $\rho = (t+1)^2$, $g = 2$, $r = t+1$, $\lambda_1 = 0$ and $\lambda_2 = 1$ (where $s = 1$, $t \geq 2$).

(v) the collection of all η - sets are given by $\nu = 2(t+1)$, $\rho = 2$, $g = t+1$, $r = 1$, $\lambda_1 = 1$ and $\lambda_2 = 0$ (where $s = 1$, $t \geq 2$).

5. Generalized Hexagon

A generalized polygon having $n = 6$ is a generalized hexsgon and is a point-line incidence structure. It consists of $(1+s)(1+st+s^2t^2)$ points and $(1+t)(1+st+s^2t^2)$ lines.

Remark 5.1. *The collection of all α, β, γ and η - sets of a generalized hexagon order (s,t) does not forms a PBIB design with m-association scheme.*

6. Generalized Octagon

A generalized polygon having $n = 8$ is a generalized octagon of order (s,t) and is a point-line incidence structure.

Remark 6.1. *The collection of all α, β, γ and η - sets of a generalized octagon order (s,t) does not forms a PBIB design with m-association scheme.*

7. Conclusion and Open problems

The main aim of this article is to initialize the study of generalized polygons of some particular cases and it gives the parameters of strongly regular graph which satisfies the parameters of the point graph of a partial geometry and also, we investigate PBIB design with association scheme for generalized digon of order s, generalized triangle, generalized quadrangle of order (s,t), generalized hexagon of order (s,t) and generalized octagon of order (s,t).

Open problem 1. Find the pseudogeometric graphs of generalized hexagon (or octagon) of order (s, t).

Open problem 2. Find the collection of all lines sets of generalized hexagon (or octagon) of order (s, t) forms a PBIB design with m-association schemes.

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