South East Asian J. of Mathematics and Mathematical Sciences
Vol. 16, No. 1 (2020), pp. 215-222

ISSN (Online): 2582-0850

ISSN (Print): 0972-7752

A NOTE ON FUZZY PRIME IDEALS OF SEMIRINGS AND Γ-SEMIRINGS

Arup Mukhopadhyay and Sarbani Mukherjee (Goswami)*

Bankura Christian College, Bankura, W.B., INDIA

E-mail : amjupm@yahoo.co.in

*Lady Brabourne College, Kolkata, W.B., INDIA

E-mail : sarbani7_goswami@yahoo.co.in

(**Received:** *Aug. 30, 2019* **Accepted:** *Mar. 10, 2020* **Published:** *Apr. 30, 2020)*

Abstract: The purpose of this paper is to study the nature of fuzzy prime ideals and fuzzy maximal ideals under semiring (resp.Γ-semiring) morphisms.

Keywords and Phrases: Semiring, Γ-semiring, Prime ideal, Fuzzy prime ideal, maximal ideal, fuzzy maximal ideal.

2010 Mathematics Subject Classification: 16Y60, 16Y99, 03E72.

1. Introduction

The notion of fuzzy set was introduced by Zadeh in 1965 [16]. This concept has been used in various branches of mathematics, since its inception. Rosenfeld [13], Kuroki [10], Jun [9] are pioneers in the field of fuzzy algebra. In 1994 [2], Dutta and Biswas introduced the notion of fuzzy prime ideal in semiring and the same was introduced by Sardar and Goswami in the Γ-semiring setting in 2010 [7]. In this paper we study the behaviour of prime ideal, fuzzy prime ideal and fuzzy maximal ideal of a semiring (Γ-semiring) under semiring (Γ-semiring) morphism.

2. Preliminaries

Here we recall some preliminary notions and relevant results in order to use them in the sequel.

Definition 2.1. [5] *Let M be a non-empty set and '+' and '.' be two binary* *operations on M, called addition and multiplication respectively. Then* $(M, +, .)$ *is called a semiring if*

(i) (M, +) is a commutative semigroup,

(ii) (M, .) is a semigroup and

(iii) $a.(b + c) = a.b + a.c$ *and* $(b + c).a = b.a + c.a$, *for all* $a, b, c \in M$.

If there exists an element $0 \in M$ such that $a + 0 = a$ for all $a \in M$ then 0 is called additive neutral element or the zero of M and M is called a semiring with zero. Moreover, if $a.0 = 0.a = 0$ for all $a \in M$ then M is called a semiring with absorbing zero.

Further if $a.b = b.a$ for all $a, b \in M$ then M is called a commutative semiring. Throughout this paper unless otherwise stated semiring means semiring with absorbing zero.

Definition 2.2. [12] *Let S and* Γ *be two additive commutative semigroups. Then S is called a* $\Gamma-$*semiring if there exists a mapping* $S \times \Gamma \times S \to S$ *(images to be denoted by* $a\alpha b$ *for* $a, b \in S$ *and* $\alpha \in \Gamma$*) satisfying the following conditions:*

(i) $(a + b)\alpha c = a\alpha c + b\alpha c$,

(ii) $a\alpha(b + c) = a\alpha b + a\alpha c$,

(iii) $a(\alpha + \beta)b = a\alpha b + a\beta b$,

(iv) $a\alpha(b\beta c) = (a\alpha b)\beta c$ *for all* $a, b, c \in S$ *and for all* $\alpha, \beta \in \Gamma$.

If $a\alpha b = b\alpha a$ *for all* $a, b \in S$ *and for all* $\alpha \in \Gamma$ *then S is said to be a commutative* $\Gamma-$*semiring.*

Definition 2.3. [5] *Let M be a semiring. A proper ideal P of M is said to be prime if for any two ideals H and K of M,* $HK \subseteq P$ *implies that either* $H \subseteq P$ *or* $K \subseteq P$.

Proposition 2.4. [5] *The following conditions on an ideal I of a semiring M are equivalent:*

(1) I is prime;

(2) $\{arb : r \in M\} \subseteq I$ *if and only if* $a \in I$ *or* $b \in I$;

(3) If a and b are elements of M satisfying $< a >< b > \subseteq I$ *then either* $a \in I$ *or* $b \in I$.

Definition 2.5. [11] *Let S be a* Γ*- semiring. A proper ideal P of S is said to be prime if for any two ideals H and K of S,* $H\Gamma K \subseteq P$ *implies that either* $H \subseteq P$ *or* $K \subseteq P$.

Definition 2.6. [2] *A fuzzy ideal* μ *of a semiring M is said to be fuzzy prime if* μ *is not a constant function and for any two fuzzy ideals* σ *and* θ *of M,* $\sigma \circ \theta \subseteq \mu$ *implies that either* $\sigma \subseteq \mu$ *or* $\theta \subseteq \mu$.

Proposition 2.7. [2] *If* μ *is a fuzzy subset of a semiring M then* μ *is a fuzzy prime* *ideal of* M *if and only if* $Im\mu = \{1, \alpha\}$ *where* $\alpha \in [0, 1)$ *and* $\mu_0 = \{x \in M : \mu(x) = \mu(0_M) = 1\}$ *is a prime ideal of* M*.*

Definition 2.8. [5] *Let* R *and* S *are semirings then a function* $\gamma : R \to S$ *is a morphism of semirings if and only if (i)* $\gamma(0_R) = 0_S$ *(ii)* $\gamma(1_R) = 1_S$ *and (iii)* $\gamma(r + r') = \gamma(r) + \gamma(r')$ *and* $\gamma(rr') = \gamma(r).\gamma(r')$ *for all* $r, r' \in R$*. A morphism of semirings which is both injective and surjective is called an isomorphism.*

Definition 2.9. [14] *A function* $f : R \to S$*, where* R, S *are* $\Gamma-$*semirings is said to be a* $\Gamma-$*morphism (or simply morphism) of* $\Gamma-$*semirings if* $f(a + b) = f(a) + f(b)$*,* $f(a\gamma b) = f(a)\gamma f(b)$ *for all* $a, b \in R$*,* $\gamma \in \Gamma$*.*

Definition 2.10. [1], [4] *Let* $f : M \to N$ *be a morphism of semirings (*$\Gamma-$*semirings).*
(i) If ϕ *is a fuzzy left ideal (fuzzy right ideal, fuzzy ideal) of* N *then* $f^{-1}(\phi)$ $[(f^{-1}(\phi))(r) := \phi(f(r))]$ *is a fuzzy left ideal (resp. fuzzy right ideal, fuzzy ideal) of* M*.*
(ii) If f *is a surjective morphism and* μ *is a fuzzy left ideal (fuzzy right ideal, fuzzy ideal) of* M *then* $f(\mu)[(f(\mu))(x) := \sup\limits_{f(y)=x} \mu(y)]$ *is a fuzzy left ideal (resp. fuzzy right ideal, fuzzy ideal) of* N*.*

Definition 2.11. [5], [15] *A proper ideal* P *of a semiring (*$\Gamma-$*semiring)* S *is said to be a maximal ideal if there is no proper ideal of* S *properly containing* P*.*

Proposition 2.12. [5], [15] *Every maximal ideal of a commutative semiring (*$\Gamma-$*semiring)* S *is prime.*

Definition 2.13. [1], [4] *Let* M *and* N *be two semirings (*Γ*-semirings) and* f *be a mapping from* M *onto* N*. A fuzzy ideal* μ *of* M *is said to be f-invariant if* $f(x) = f(y)$ *implies that* $\mu(x) = \mu(y)$ *for all* $x, y \in M$*.*

3. Nature of a fuzzy prime ideal under a semiring morphism

Proposition 3.1. *If* $f : M \to N$ *is a morphism of semirings and* μ *is* $f-$*invariant fuzzy ideal of* M*, then*
(i) $[f(\mu)](0_N) = \mu(0_M)$*,*
(ii)$f(\mu_0) = [f(\mu)]_0$*.*
Proof. (i) $[f(\mu)](0_N) = \sup\limits_{f(x)=0_N} \mu(x) = \sup\limits_{f(x)=f(0_M)} \mu(x) = \sup\limits_{f(x)=f(0_M)} \mu(0_M)$
$= \mu(0_M)$, since μ is $f-$invariant.
(ii) Let $y \in f(\mu_0)$. Then $y = f(x)$ for some $x \in \mu_0$. Now $[f(\mu)](y)$
$= \sup\limits_{f(z)=y} \mu(z) = \sup\limits_{f(z)=f(x)} \mu(z) = \mu(x) = \mu(0_M) = [f(\mu)](0_N)$ by (i). Hence $y \in [f(\mu)]_0$. So $f(\mu_0) \subseteq [f(\mu)]_0$. Again let $f(x) \in [f(\mu)]_0$. Then

$[f(\mu)](0_N) = [f(\mu)](f(x)) = \sup_{f(t)=f(x)} \mu(t) = \mu(x)$. i.e., $\mu(0_M) = \mu(x)$.
So $x \in \mu_0$ and consequently, $f(x) \in f(\mu_0)$. Thus $[f(\mu)]_0 \subseteq f(\mu_0)$ and hence $f(\mu_0) = [f(\mu)]_0$.

Proposition 3.2. *If $f : M \to N$ is a surjective morphism of semirings and P be a prime ideal of M, then the ideal $f(P)$ of N is prime.*
Proof. Let $f(a)Nf(b) \subseteq f(P)$ where $a, b \in M$. i.e., $f(a)f(M)f(b) \subseteq f(P)$. Then for an arbitrary $s \in M$, $f(a)f(s)f(b) \in f(P)$ which implies that $f(asb) \in f(P)$ and so $asb \in P$. Since $s \in M$ is arbitrary, we have $aMb \subseteq P$ Therefore either $a \in P$ or $b \in P$ and hence either $f(a) \in f(P)$ or $f(b) \in f(P)$. Consequently, $f(P)$ is a prime ideal of N.

Proposition 3.3. *If $f : M \to N$ is a surjective morphism of semirings and P be a maximal ideal of M then the ideal $f(P)$ of N is maximal.*
Proof. Let P be a maximal ideal in M. Then $f(P)$ is a ideal of N. If possible let $f(P)$ is not maximal. Then there exist an ideal Q in N such that $f(P) \subseteq Q$. Implies that $P \subseteq f^{-1}(Q)$, an ideal of M, which contradicts that P is maximal in M. Hence $f(P)$ is maximal in N.

Theorem 3.4. *Let f be a morphism of a semiring M onto a semiring N. If μ is an $f-$invariant fuzzy prime ideal of M, then $f(\mu)$ is a fuzzy prime ideal of N.*
Proof. Let μ be an $f-$invariant fuzzy prime ideal of M. Then by Proposition 2.10, $f(\mu)$ is a fuzzy ideal of N. Now since μ is fuzzy prime, by Proposition 2.7, (i) $\mu(0_M) = 1$, (ii) $Im\mu = \{1, \alpha\}$ with $\alpha \in [0, 1)$,
(iii) $\mu_0 = \{x \in M : \mu(x) = \mu(0_M)\}$ is a prime ideal of M. Now by Proposition 3.1 and Proposition 3.2, $[f(\mu)](0_N) = \mu(0_M) = 1$ and $[f(\mu)]_0 = f(\mu_0)$ is a prime ideal of M. We shall now prove that $[f(\mu)](N) = \{1, \alpha\}$, $\alpha \in [0, 1)$. Let $x \in M$ be such that $\mu(x) = \alpha$. Then $[f(\mu)](f(x)) = \sup_{f(z)=f(x)} \mu(z) = \mu(x) = \alpha$, as μ is $f-$invariant. Again $[f(\mu)](0_N) = 1$, thus $[f(\mu)](N) = \{1, \alpha\}$ and consequently $f(\mu)$ is a fuzzy prime ideal of N (*cf. Proposition 2.7*).

Definition 3.5. *A non-constant fuzzy ideal μ of a semiring M is called fuzzy maximal ideal if $Im\ \mu = \{1, \alpha\}$ where $\alpha \in [0, 1)$ and the level ideal $\mu_0 = \{x \in M : \mu(x) = 1\}$ is a maximal ideal of M.*

Proposition 3.6. *Every fuzzy maximal ideal of a commutative semiring M is fuzzy prime ideal of M.*
Proof. Let μ be a fuzzy maximal ideal of a commutative semiring M. Then (i) $Im\mu = \{1, \alpha\}$ with $\alpha \in [0, 1)$, and (ii) μ_0 is a maximal ideal of M. Since every maximal ideal of a commutative semiring is prime, so μ_0 is prime ideal of M. Hence

by Proposition 2.7, μ is fuzzy prime ideal of M.

The converse of the above proposition is not be true in general. For example let us consider the following:

Example 3.7. Let S be the set of all non-negative integers. Then $(S, +, .)$ forms a semiring under usual addition and multiplication of integers. Let us consider the ideal $I = \{0\}$ of S. We define a fuzzy ideal μ of S as follows:

$$\mu(x) = \begin{cases} 1 & \text{if } \ x \in I \\ 0 & \text{if } x \notin I \end{cases}$$

Then μ is a fuzzy prime ideal of S. But not a fuzzy maximal ideal of S as $\mu_0 = I$ is not a maximal ideal of S.

Again it may be noted that the above proposition may not be true if we drop the condition that the semiring S is commutative. Consider the following example in this regard.

Example 3.8. Let R^+ be the set of all non-negative reals. Then $(M_2(R^+), +, .)$ forms a non-commutative semiring under usual addition and multiplication of matrices. Let us consider the ideal $I = \left\{ \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \right\}$ of $M_2(R^+)$. We define a fuzzy ideal μ of $M_2(R^+)$ as follows:

$$\mu(x) = \begin{cases} 1 & \text{if } \ x \in I \\ 0 & \text{if } x \notin I \end{cases}$$

Then μ is a fuzzy maximal ideal of $M_2(R^+)$ as $\mu_0 = I$ is the only proper ideal of $M_2(R^+)$ and hence maximal ideal of $M_2(R^+)$. But μ is not a fuzzy prime ideal of $M_2(R^+)$.

Proposition 3.9. *Let f be a morphism of a semiring M onto a semiring N. If μ is a f-invariant fuzzy maximal ideal of M, then $f(\mu)$ is a fuzzy maximal ideal of N.*
Proof. Let μ be an $f-$invariant fuzzy maximal ideal of M. Then by Proposition 2.10, $f(\mu)$ is a fuzzy ideal of N. Now since μ is fuzzy maximal, then by, Definition 3.5.(i) $Im\mu = \{1, \alpha\}$ with $\alpha \in [0, 1)$,
(ii) $\mu_0 = \{x \in M : \mu(x) = \mu(0_M)\}$ is a maximal ideal of M. By a similar argument as in the proof of Theorem 3.4, we have $Imf(\mu) = \{1, \alpha\}, \ \alpha \in [0, 1)$. Again since μ_0 is a maximal ideal of M, we have $[f(\mu)]_0 = f(\mu_0)$, is also a maximal ideal of N by Proposition 3.3. Hence $f(\mu)$ is a fuzzy maximal ideal of N.

Proposition 3.10. *Let f be a morphism from a semiring M to a semiring N and let μ be a fuzzy ideal of N. Then $f^{-1}(\mu_0) = [f^{-1}(\mu)]_0$.*

Proof. Let $x \in M$. Now,

$$\begin{aligned} x \in f^{-1}(\mu_0) &\Leftrightarrow f(x) \in \mu_0 \\ &\Leftrightarrow \mu(f(x)) = \mu(0_N) = \mu(f(0_M)) \\ &\Leftrightarrow f^{-1}(\mu)(x) = f^{-1}(\mu)(0_M) \\ &\Leftrightarrow x \in [f^{-1}(\mu)]_0 \end{aligned}$$

Thus $f^{-1}(\mu_0) = [f^{-1}(\mu)]_0$.

Proposition 3.11. *Let f be a morphism from a semiring M onto a semiring N and let P be a prime ideal of N. Then $f^{-1}(P)$ is a prime ideal of M.*
Proof. Let $aMb \subseteq f^{-1}(P)$. Then for $s \in M$, we have $asb \in f^{-1}(P)$ $\Rightarrow f(asb) \in P \Rightarrow f(a)f(s)f(b) \in P$ for all $s \in S$. Therefore $f(a)f(M)f(b) \subseteq P$. Thus $f(a)Nf(b) \subseteq P$, as f is onto. This implies that either $f(a) \in P$ or $f(b) \in P$ as P is a prime ideal of N whence either $a \in f^{-1}(P)$ or $b \in f^{-1}(P)$. Thus $f^{-1}(P)$ is a fuzzy prime ideal of M.

Proposition 3.12. *Let f be a morphism from a semiring M onto a semiring N and let P be a maximal ideal of N. Then $f^{-1}(P)$ is a maximal ideal of M.*
Proof. Let P be a maximal ideal in N. Implies that $f^{-1}(P)$ is an ideal of M. If possible let $f^{-1}(P)$ is not maximal ideal of M, then there exist an ideal Q in M such that $f^{-1}(P) \subseteq Q$. Implies that $P \subseteq f(Q)$, which is an ideal of N. This contradicts the maximality of P in N. Hence $f^{-1}(P)$ is a maximal ideal of M.

Theorem 3.13. *Let f be a morphism from a semiring M onto a semiring N and let μ be a fuzzy prime ideal of N. Then $f^{-1}(\mu)$ is a fuzzy prime ideal of M. Moreover, $f^{-1}(\mu)$ is f-invariant.*
Proof. Let f be a morphism from a semiring M onto a semiring N and let μ be a fuzzy prime ideal of N. By proposition 2.10, $f^{-1}(\mu)$ is a fuzzy ideal of M. Now since μ is a fuzzy prime ideal of N we have,
(i) $\mu(0_N) = 1$,
(ii) $Im\mu = \{1, \alpha\}$ with $\alpha \in [0, 1)$,
(iii) $\mu_0 = \{x \in N : \mu(x) = \mu(0_N)\}$ is a prime ideal of N.
Thus $[f^{-1}(\mu)](0_M) = \mu(f(0_M)) = \mu(0_N) = 1$, using (i). Next let $y \in N$ be such that $\mu(y) = \alpha$, then there exists $x \in M$ such that $f(x) = y$ as f is onto. Now $f^{-1}(\mu)(x) = \mu(f(x)) = \alpha$. Therefore $f^{-1}(\mu)(M) = \{1, \alpha\},\ \alpha \in [0, 1)$. $[f^{-1}(\mu)]_0 = f^{-1}(\mu_0)$ is a prime ideal of M. Hence $f^{-1}(\mu)$ is a fuzzy prime ideal of M. Further, let $f(x) = f(y)$ then $(f^{-1}(\mu))(x) = \mu(f(x)) = \mu(f(y)) = (f^{-1}(\mu))(y)$. Thus $f^{-1}(\mu)$ is f-invariant.

Proposition 3.14. *Let f be a morphism from a semiring M onto a semiring N*

and let μ be a fuzzy maximal ideal of N. Then $f^{-1}(\mu)$ is a fuzzy maximal ideal of M.

Proof. Let f be a morphism from a semiring M onto a semiring N and let μ be a fuzzy maximal ideal of N. By Proposition 2.10, $f^{-1}(\mu)$ is a fuzzy ideal of M. Now since μ is a fuzzy maximal ideal of N we have,

(i) $Im\mu = \{1, \alpha\}$ with $\alpha \in [0, 1)$,

(ii) $\mu_0 = \{x \in N : \mu(x) = \mu(0_N)\}$ is a maximal ideal of N.

Now, by similar argument as the proof of the Theorem 3.13 we have $Imf^{-1}(\mu) = \{1, \alpha\}$ with $\alpha \in [0, 1)$. Again μ_0 is a maximal ideal of N. Implies that $[f^{-1}(\mu)]_0 = f^{-1}(\mu_0)$ is a maximal ideal of M by Proposition 3.10. Hence $f^{-1}(\mu)$ is a fuzzy maximal ideal of M.

Remark 3.15. *The analogues of the above results in the setting of $\Gamma-$semirings are also valid. Details of which we omit because proofs are, to some extent, similar.*

Acknowledgement

We are thankful to Prof. Sujit Kr Sardar, Department of Mathematics, Jadavpur University for his constant guidance throughout the preparation of this paper. We are also thankful to the respected Referee for his valuable suggestions and comments.

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