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STUDY ON GROWTH OF k-ITERATED ENTIRE FUNCTIONS

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Abstract: Considering k entire functions, we study growth of k-iterated entire functions to generalise some earlier results.

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1. Introduction

Let $f(z)$ and $g(z)$ be two transcendental entire functions defined in $\mathbb{C}$. We know [2] that $\lim_{r\to\infty}\frac{T(r,f\circ g)}{T(r,f)}=\infty$ and $\lim_{r\to\infty}\frac{T(r,f\circ g)}{T(r,g)}=\infty$. Lahiri and Datta [7] investigated comparative growth properties of $logT(r,f\circ g)$ and $T(r,g)$ together with that of $loglogT(r,f\circ g)$ and $T\left(r,f^{(l)}\right)$. After this, Banerjee and Dutta [1] considering two functions $f(z)$ and $g(z)$ and following Lahiri and Banerjee [4] formed relative iterations and studied the growth properties of iterated entire functions.

In this paper we consider k entire functions $f_1(z), f_2(z), f_3(z), ..., f_k(z)$ and form the iteration [defined below] to generalise the results of Banerjee and Dutta [1].

Let

$$\begin{aligned}
F_1^1(z) &= f_1(z)\\
F_2^1(z) &= f_1(f_2(z)) = f_1\left(F_1^2(z)\right)\\
F_3^1(z) &= f_1(f_2(f_3(z))) = f_1\left(f_2\left(F_1^3(z)\right)\right) = f_1(F_2^2(z))\\
&\vdots\\
F_k^1(z) &= f_1(f_2(...f_k(z))) = f_1\left(F_{k-1}^2(z)\right) = ... = f_1\left(f_2...f_{k-1}\left(F_1^k(z)\right)\right)\\
&\vdots\\
F_n^1(z) &= f_1(f_2(f_3...(f_1(z) \text{ or } f_2(z) \text{ or...or } f_k(z) \text{ according as}\\
&\quad n = km-(k-1) \text{ or } km-(k-2) \text{ or ...or } km)...)) = f_1\left(F_{n-1}^2(z)\right)\\
&\quad = ... = f_1\left(f_2\left(...\left(f_{k-1}\left(F_{n-(k-1)}^k(z)\right)\right)\right)\right).
\end{aligned}$$

Similarly

$$\begin{aligned}
F_1^2(z) &= f_2(z)\\
F_2^2(z) &= f_2(f_3(z)) = f_2\left(F_1^3(z)\right)\\
F_3^2(z) &= f_2(f_3(f_4(z))) = f_2\left(f_3\left(F_1^4(z)\right)\right) = f_2(F_2^2(z))\\
&\vdots\\
F_k^2(z) &= f_2(f_3(...f_k(f_1(z)))) = f_2\left(F_{k-1}^3(z)\right) = ... = f_2\left(f_3\left(...f_k\left(F_1^1(z)\right)\right)\right)\\
&\vdots\\
F_n^2(z) &= f_2(f_3(f_4...(f_2(z) \text{ or } f_3(z) \text{ or ...or } f_k(z) \text{ or } f_1(z) \text{ according as}\\
&\quad n = km-(k-1) \text{ or } km-(k-2) \text{ or ...or } (km-1) \text{ or } km)...))\\
&\quad = f_2\left(F_{n-1}^3(z)\right) = ... = f_2\left(f_3\left(...\left(f_k\left(F_{n-(k-1)}^1(z)\right)\right)\right)\right).
\end{aligned}$$

And

$$\begin{aligned}
F_1^k(z) &= f_k(z)\\
F_2^k(z) &= f_k(f_1(z)) = f_k\left(F_1^1(z)\right)\\
F_3^k(z) &= f_k(f_1(f_2(z))) = f_k\left(f_1\left(F_1^2(z)\right)\right) = f_k(F_2^1(z))
\end{aligned}$$

$$\begin{aligned}
&\vdots\\
F_k^k(z) &= f_k(f_1(...f_{k-2}(f_{k-1}(z)))) = ... = f_k\left(f_1\left(...f_{k-2}\left(F_1^{k-1}(z)\right)\right)\right)\\
&\vdots\\
F_n^k(z) &= f_k(f_1(f_2...(f_k(z) \text{ or } f_1(z) \text{ or } ...\text{or } f_{k-1}(z) \text{ according as}\\
&\quad n = km-(k-1) \text{ or } km-(k-2) \text{ or } ...\text{or } km)...)) = f_k\left(F_{n-1}^1(z)\right)\\
&= .... = f_k\left(f_1\left(...\left(f_{k-2}\left(F_{n-(k-1)}^{k-1}(z)\right)\right)\right)\right).
\end{aligned}$$

Here all $F_n^1, F_n^2, ..., F_n^k$ are entire functions.

For two non-constant entire functions $f(z)$ and $g(z)$, the inequality

$$\log M(r, f(g)) \le \log M(M(r, g), f) \tag{1}$$

obviously holds. Also we need the following definitions.

Definition 1.1. *The order ρ_f and lower order λ_f of a meromorphic function is defined as*

$$\rho_f = \limsup_{r\to\infty} \frac{\log T(r,f)}{\log r}$$

and

$$\lambda_f = \liminf_{r\to\infty} \frac{\log T(r,f)}{\log r}.$$

If f is entire then

$$\rho_f = \limsup_{r\to\infty} \frac{\log\log M(r,f)}{\log r}$$

and

$$\lambda_f = \liminf_{r\to\infty} \frac{\log\log M(r,f)}{\log r}.$$

Definition 1.2. *The hyper order $\overline{\rho_f}$ and hyper lower order $\overline{\lambda}_f$ of a meromorphic function is defined as*

$$\overline{\rho_f} = \limsup_{r\to\infty} \frac{\log\log T(r,f)}{\log r}$$

and

$$\overline{\lambda}_f = \liminf_{r\to\infty} \frac{\log\log T(r,f)}{\log r}.$$

If f is entire then

$$\overline{\rho_f} = \limsup_{r\to\infty} \frac{\log^{[3]} M(r,f)}{\log r}$$

and

$$\overline{\lambda}_f = \liminf_{r\to\infty} \frac{\log^{[3]} M(r,f)}{\log r}.$$

Definition 1.3. *A function $\lambda_f(r)$ is called a lower proximate order of a meromorphic function f if*

(i) $\lambda_f(r)$ is nonnegative and continuous for $r \geq r_0$, say;

(ii) $\lambda_f(r)$ is differentiable for $r \geq r_0$ except possibly at isolated points at which $\lambda_f^{'}(r-0)$ and $\lambda_f^{'}(r+0)$ exist;

(iii) $\lim_{r\to\infty} \lambda_f(r) = \lambda_f < \infty$;

(iv) $\lim_{r\to\infty} r\lambda_f^{'}(r)\log r = 0$; *and*

(v) $\liminf_{r\to\infty} \frac{T(r,f)}{r^{\lambda_f(r)}} = 1.$

Notation 1.4. Following Sato [10] we write $log^{[0]}x = x$, $exp^{[0]}x = x$ and $log^{[m]}x = log(log^{[m-1]}x)$, $exp^{[m]}x = exp(exp^{[m-1]}x)$ for $m \in \mathbb{N}$.

Throughout we assume $f_1, f_2, ..., f_k$ etc. are non constant entire functions having respective orders $\rho_{f_1}, \rho_{f_2}, ..., \rho_{f_k}$ and respective lower orders $\lambda_{f_1}, \lambda_{f_2}, ..., \lambda_{f_k}$.

2.Lemmas

The following lemmas will be needed.

Lemma 2.1. [3] *Let $f(z)$ be an entire function. For $0 \leq r < R < \infty$, we have*

$$T(r,f) \leq \log^+ M(r,f) \leq \frac{R+r}{R-r} T(R,f).$$

Lemma 2.2. [9] *Let $f_1(z)$ and $f_2(z)$ be two entire functions. Then we have*

$$T(r, f_1(f_2)) \geq \frac{1}{3}\log M\left(\frac{1}{8}M\left(\frac{r}{4}, f_2\right) + O(1), f_1\right).$$

Lemma 2.3. [5] *Let f be an entire function. Then for $k > 2$,*

$$\liminf_{r\to\infty} \frac{\log^{[k-1]} M(r,f)}{\log^{[k-2]} T(r,f)} = 1.$$

Lemma 2.4. [7] *Let f be a meromorphic function. Then for $\delta(>0)$ the function $r^{\lambda_f + \delta - \lambda_f(r)}$ is an increasing function of r.*

Lemma 2.5. [8] *Let f be an entire function of finite lower order. If there exist entire functions a_i $(i = 1, 2, 3..........n;\ n \leq \infty)$ satisfying $T(r, a_i) = o\{T(r,f)\}$ and*

$$\sum_{i=1}^{n} \delta(a_i, f) = 1 \quad then \quad \lim_{r\to\infty} \frac{T(r,f)}{\log M(r,f)} = \frac{1}{\pi}.$$

Lemma 2.6. *Let $f_1(z)$, $f_2(z)$,..., $f_k(z)$ be k entire functions such that $0 < \lambda_{f_i} \leq \rho_{f_i} < \infty$; $i = 1, 2, ..., k$. Then for any ε $(0 < \varepsilon < min\{\lambda_{f_1}, \lambda_{f_2}, ..., \lambda_{f_k}\})$ and for all large r we have*

$$\log^{[n-1]} T(r, F_n^1) \leq \begin{cases} (\rho_{f_{k-1}} + \varepsilon) \log M(r, f_k) + O(1) \text{ when } n = km, \ m \in \mathbb{N} \\ (\rho_{f_{k-2}} + \varepsilon) \log M(r, f_{k-1}) + O(1) \text{ when } n = km - 1, \ m \in \mathbb{N} \\ \vdots \\ (\rho_{f_k} + \varepsilon) \log M(r, f_1) + O(1) \text{ when } n = km - (k-1), \ m \in \mathbb{N} \end{cases}$$

and

$$\log^{[n-1]} T(r, F_n^1) \geq \begin{cases} (\lambda_{f_{k-1}} - \varepsilon) \log M\left(\frac{r}{4^{n-1}}, f_k\right) + O(1) \text{ when } n = km, \ m \in \mathbb{N} \\ (\lambda_{f_{k-2}} - \varepsilon) \log M\left(\frac{r}{4^{n-1}}, f_{k-1}\right) + O(1) \text{ when } n = km - 1, \ m \in \mathbb{N} \\ \vdots \\ (\lambda_{f_k} - \varepsilon) \log M\left(\frac{r}{4^{n-1}}, f_1\right) + O(1) \text{ when } n = km - (k-1), \ m \in \mathbb{N}. \end{cases}$$

Proof. For $\varepsilon (> 0)$ from Lemma 2.1 and (1) for all large values of r

$$\begin{aligned} T(r, F_n^1) &\leq \log M(r, F_n^1) \\ &\leq \log M(M(r, F_{n-1}^2), f_1) \\ &\leq [M(r, F_{n-1}^2)]^{\rho_{f_1} + \varepsilon} \\ \text{i.e. , } \log T(r, F_n^1) &\leq (\rho_{f_1} + \varepsilon) \log M(r, F_{n-1}^2) \\ &\leq (\rho_{f_1} + \varepsilon) \log M(M(r, F_{n-2}^3), f_2) \\ &\leq (\rho_{f_1} + \varepsilon)[M(r, F_{n-2}^3)]^{\rho_{f_2} + \varepsilon}. \\ \text{Therefore, } \log^{[2]} T(r, F_n^1) &\leq (\rho_{f_2} + \varepsilon) \log M(M(r, F_{n-3}^4), f_3) + O(1) \\ &\leq (\rho_{f_2} + \varepsilon)[M(r, F_{n-3}^4)]^{\rho_{f_3} + \varepsilon} + O(1). \\ \text{So, } \log^{[3]} T\left(r, F_n^1\right) &\leq (\rho_{f_3} + \varepsilon) \log M\left(M\left(r, F_{n-4}^5\right), f_4\right) + O(1) \\ &\leq (\rho_{f_3} + \varepsilon)\left[M\left(r, F_{n-4}^5\right)\right]^{\rho_{f_4} + \varepsilon} + O(1). \\ \text{Therefore, } \log^{[n-1]} T(r, F_n^1) &\leq (\rho_{f_{k-1}} + \varepsilon) \log M(r, f_k) + O(1) \text{when } n = km, \ m \in \mathbb{N}. \end{aligned}$$

Similarly,

$$\log^{[n-1]} T(r, F_n^1) \leq (\rho_{f_{k-2}} + \varepsilon) \log M(r, f_{k-1}) + O(1) \quad \text{when } n = km - 1, \ m \in \mathbb{N}$$

$$\vdots$$

$$\text{and } \log^{[n-1]} T\left(r, F_n^1\right) \leq (\rho_{f_k} + \varepsilon) \log M(r, f_1) + O(1) \quad \text{when } n = km - (k-1), \ m \in \mathbb{N}.$$

Again for ε $(0<\varepsilon<\min\{\lambda_{f_1},\lambda_{f_2},...,\lambda_{f_k}\})$, we have from Lemma 2.1 and Lemma 2.2 for all large values of r

$$\begin{aligned}
T(r,F_n^1) &= T(r,f_1(F_{n-1}^2)) \\
&\geq \frac{1}{3}\log M\left(\frac{1}{8}M\left(\frac{r}{4},F_{n-1}^2\right)+O(1),f_1\right) \\
&\geq \frac{1}{3}\left[\frac{1}{9}M\left(\frac{r}{4},F_{n-1}^2\right)\right]^{\lambda_{f_1}-\varepsilon}, \\
\text{i.e. , } \log T(r,F_n^1) &\geq (\lambda_{f_1}-\varepsilon)\log M\left(\frac{r}{4},F_{n-1}^2\right)+O(1) \\
&\geq (\lambda_{f_1}-\varepsilon)T\left(\frac{r}{4},F_{n-1}^2\right)+O(1) \\
&\geq (\lambda_{f_1}-\varepsilon)\frac{1}{3}\log M\left(\frac{1}{8}M\left(\frac{r}{4^2},F_{n-2}^3\right)+O(1),f_2\right)+O(1) \\
&\geq (\lambda_{f_1}-\varepsilon)\frac{1}{3}\left[\frac{1}{9}M\left(\frac{r}{4^2},F_{n-2}^3\right)\right]^{\lambda_{f_2}-\varepsilon}+O(1). \\
\text{So , } \log^{[2]}T(r,F_n^1) &\geq (\lambda_{f_2}-\varepsilon)\log M\left(\frac{r}{4^2},F_{n-2}^3\right)+O(1) \\
\text{i.e., } \log^{[3]}T(r,F_n^1) &\geq (\lambda_{f_3}-\varepsilon)\log M\left(\frac{r}{4^3},F_{n-3}^4\right)+O(1). \\
\text{Then } \log^{[n-2]}T\left(r,F_n^1\right) &\geq \left(\lambda_{f_{k-2}}-\varepsilon\right)\log M\left(\frac{r}{4^{n-2}},f_{k-1}\circ f_k\right). \qquad (2) \\
\text{So, } \log^{[n-1]}T(r,F_n^1) &\geq (\lambda_{f_{k-1}}-\varepsilon)\log M\left(\frac{r}{4^{n-1}},f_k\right)+O(1)\text{when } n=km, m\in\mathbb{N}.
\end{aligned}$$

Similarly

$$\begin{aligned}
\log^{[n-1]}T(r,F_n^1) &\geq (\lambda_{f_{k-2}}-\varepsilon)\log M\left(\frac{r}{4^{n-1}},f_{k-1}\right)+O(1) \\
&\qquad \text{when } n=km-1, m\in\mathbb{N} \\
&\quad\vdots \\
\text{and } \log^{[n-1]}T\left(r,F_n^1\right) &\geq (\lambda_{f_k}-\varepsilon)\log M\left(\frac{r}{4^{n-1}},f_1\right)+O\left(1\right) \\
&\qquad \text{when } n=km-(k-1), m\in\mathbb{N}.
\end{aligned}$$

This completes the proof.

3. Main Results

In this section we present the main results of the paper.

Theorem 3.1. *Let $f_1,f_2,...,f_k$ be k entire functions having positive lower orders.*

Then

$$\liminf_{r\to\infty} \frac{\log^{[n-1]} T(r, F_n^1)}{T(r, f_k)} \leq 3\rho_{f_{k-1}} 2^{\lambda_{f_k}},$$
$$\limsup_{r\to\infty} \frac{\log^{[n-1]} T(r, F_n^1)}{T(r, f_k)} \geq \frac{\lambda_{f_{k-1}}}{(4^{n-1})^{\lambda_{f_k}}}$$

when $n = km$, $m \in \mathbb{N}$.

Proof. Assuming $0 < \lambda_{f_k} \leq \rho_{f_k} < \infty$, we have from Lemma 2.6 for arbitrary $\varepsilon > 0$

$$\log^{[n-1]}\ T(r, F_n^1) \leq (\rho_{f_{k-1}} + \varepsilon) \log M(r, f_k) + O(1) \tag{3}$$

when $n = km$, $m \in \mathbb{N}$.

Let $0 < \varepsilon <\min\{1, \lambda_{f_1}, \lambda_{f_2}, ..., \lambda_{f_k}\}$. Again since

$$\liminf_{r\to\infty} \frac{T(r, f_k)}{r^{\lambda_{f_k}(r)}} = 1,$$

so there is a sequence of values of r tending to infinity for which

$$T(r, f_k) < (1 + \varepsilon) r^{\lambda_{f_k}(r)}. \tag{4}$$

Also for all large value of r

$$T(r, f_k) > (1 - \varepsilon) r^{\lambda_{f_k}(r)}. \tag{5}$$

So we have for a sequence of values of r tending to infinity and for any $\delta(> 0)$

$$\begin{aligned} \frac{\log M(r, f_k)}{T(r, f_k)} &\leq \frac{3T(2r, f_k)}{T(r, f_k)} \leq \frac{3(1+\varepsilon)}{1-\varepsilon} \frac{(2r)^{\lambda_{f_k}+\delta}}{(2r)^{\lambda_{f_k}+\delta-\lambda_{f_k}(2r)}} \frac{1}{r^{\lambda_{f_k}(r)}} \\ &\leq \frac{3(1+\varepsilon)}{1-\varepsilon} 2^{\lambda_{f_k}+\delta} \end{aligned}$$

as $r^{\lambda_{f_k}+\delta-\lambda_{f_k}(r)}$ is an increasing function of r.

Since ε, $\delta > 0$ be arbitrary, so we have

$$\liminf_{r\to\infty} \frac{\log M(r, f_k)}{T(r, f_k)} \leq 3.2^{\lambda_{f_k}}. \tag{6}$$

Therefore from (3) and (6) we get

$$\liminf_{r\to\infty} \frac{\log^{[n-1]} T(r, F_n^1)}{T(r, f_k)} \leq 3\rho_{f_{k-1}} 2^{\lambda_{f_k}}$$

when $n = km$, $m \in \mathbb{N}$.
Again from Lemma 2.6 for $n = km$, $m \in \mathbb{N}$ we have

$$\begin{aligned}\log^{[n-1]} T(r, F_n^1) &\geq (\lambda_{f_{k-1}} - \varepsilon) \log M\left(\frac{r}{4^{n-1}}, f_k\right) + O(1) \\ &\geq (\lambda_{f_{k-1}} - \varepsilon) T\left(\frac{r}{4^{n-1}}, f_k\right) + O(1) \\ &\geq (\lambda_{f_{k-1}} - \varepsilon)(1-\varepsilon)(1+O(1)) \frac{\left(\frac{r}{4^{n-1}}\right)^{\lambda_{f_k}+\delta}}{\left(\frac{r}{4^{n-1}}\right)^{\lambda_{f_k}+\delta-\lambda_{f_k}\left(\frac{r}{4^{n-1}}\right)}}, \text{ by (5).}\end{aligned}$$

Again since $r^{\lambda_{f_k}+\delta-\lambda_{f_k}(r)}$ is an increasing function of r, we have

$$\log^{[n-1]} T(r, F_n^1) \geq (\lambda_{f_{k-1}} - \varepsilon)(1-\varepsilon)(1+O(1)) \frac{r^{\lambda_{f_k}(r)}}{(4^{n-1})^{\lambda_{f_k}+\delta}}$$

for all large values of r.

Therefore by (4) for a sequence of values of r tending to infinity

$$\log^{[n-1]} T(r, F_n^1) \geq (\lambda_{f_{k-1}} - \varepsilon) \frac{1-\varepsilon}{1+\varepsilon}(1+O(1)) \frac{T(r, f_k)}{(4^{n-1})^{\lambda_{f_k}+\delta}}.$$

Since ε and δ are arbitrary, so

$$\limsup_{r\to\infty} \frac{\log^{[n-1]} T(r, F_n^1)}{T(r, f_k)} \geq \frac{\lambda_{f_{k-1}}}{(4^{n-1})^{\lambda_{f_k}}}.$$

This completes the proof.

Corollary 3.2. *Let $f_1, f_2, ..., f_k$ be k entire functions having finite lower orders. Then*

$$\begin{aligned}\liminf_{r\to\infty} \frac{\log^{[n-1]} T(r, F_n^1)}{T(r, f_1)} &\leq 3\rho_{f_k} 2^{\lambda_{f_1}}, \\ \limsup_{r\to\infty} \frac{\log^{[n-1]} T(r, F_n^1)}{T(r, f_1)} &\geq \frac{\lambda_{f_k}}{(4^{n-1})^{\lambda_{f_1}}}\end{aligned}$$

when $n = km - (k-1)$, $m \in \mathbb{N}$.

Theorem 3.3. *Let $f_1, f_2, ..., f_k$ be k entire functions having positive lower orders. Also there exist entire functions a_i ($i = 1, 2, 3..........n$; $n \leq \infty$) satisfying $T(r, a_i) = o\{T(r, f_k)\}$ as $r \to \infty$ and*

$$\sum_{i=1}^{n} \delta(a_i, f_k) = 1.$$

Then

$$\frac{\pi\lambda_{f_{k-1}}}{(4^{n-1})^{\lambda_{f_k}}} \leq \limsup_{r\to\infty} \frac{\log^{[n-1]} T(r, F_n^1)}{T(r, f_k)} \leq \pi\rho_{f_{k-1}}$$

when $n = km$, $m \in \mathbb{N}$.

Proof. For $0 < \varepsilon < \min\{1, \lambda_{f_1}, \lambda_{f_2}, ..., \lambda_{f_k}\}$ we have from Lemma 2.6. for all large values of r

$$\begin{aligned} \frac{\log^{[n-1]}\ T(r, F_n^1)}{T(r, f_k)} &\geq (\lambda_{f_{k-1}} - \varepsilon)\frac{\log M\left(\frac{r}{4^{n-1}}, f_k\right)}{T(r, f_k)} + O(1), \quad \text{where } n = km \\ &= (\lambda_{f_{k-1}} - \varepsilon)\frac{\log M\left(\frac{r}{4^{n-1}}, f_k\right)}{T\left(\frac{r}{4^{n-1}}, f_k\right)}\frac{T\left(\frac{r}{4^{n-1}}, f_k\right)}{T(r, f_k)} + O(1). \end{aligned} \tag{7}$$

From (4) and (5) for a sequence of values of $r \to \infty$ and for $\delta > 0$ we get

$$\begin{aligned} \frac{T\left(\frac{r}{4^{n-1}}, f_k\right)}{T(r, f_k)} &> \frac{1-\varepsilon}{1+\varepsilon}\frac{\left(\frac{r}{4^{n-1}}\right)^{\lambda_{f_k}+\delta}}{\left(\frac{r}{4^{n-1}}\right)^{\lambda_{f_k}+\delta-\lambda_{f_k}\left(\frac{r}{4^{n-1}}\right)}}\frac{1}{r^{\lambda_{f_k}(r)}} \\ &\geq \frac{1-\varepsilon}{1+\varepsilon}\frac{1}{(4^{n-1})^{\lambda_{f_k}+\delta}} \end{aligned}$$

as $r^{\lambda_{f_k}+\delta-\lambda_{f_k}(r)}$ is an increasing function of r.

Again $\varepsilon,\ \delta > 0$ be arbitrary, so using Lemma 2.5, we have from (7)

$$\limsup_{r\to\infty} \frac{\log^{[n-1]} T(r, F_n^1)}{T(r, f_k)} \geq \frac{\pi\lambda_{f_{k-1}}}{(4^{n-1})^{\lambda_{f_k}}}.$$

If $\rho_{f_{k-1}} = \infty$, the second inequality is obvious. Let $\rho_{f_{k-1}} < \infty$. Therefore the second inequality follows from Lemma 2.5 and Lemma 2.6.

This completes the proof.

Corollary 3.4. *Let* $f_1, f_2, ..., f_k$ *be* k *entire functions such that* $\lambda_{f_1}, \lambda_{f_2}, ..., \lambda_{f_k} (> 0)$ *are finite. Also there exist entire functions* a_i $(i = 1, 2, 3..........n;\ n \leq \infty)$ *satisfying* $T(r, a_i) = o\{T(r, f_1)\}$ *as* $r \to \infty$ *and*

$$\sum_{i=1}^{n} \delta(a_i, f_1) = 1.$$

Then

$$\frac{\pi\lambda_{f_k}}{(4^{n-1})^{\lambda_{f_1}}} \leq \limsup_{r\to\infty} \frac{\log^{[n-1]} T(r, F_n^1)}{T(r, f_1)} \leq \pi\rho_{f_k}$$

when $n = km - (k-1)$, $m \in \mathbb{N}$.

Theorem 3.5. *Let* $f_1, f_2, ..., f_k$ *be* k *entire functions such that* $0 < \lambda_{f_i} \le \rho_{f_i} < \infty, i = 1, 2, ..., k$. *Then for* $l = 0, 1, 2, 3, ...$

$$\frac{\overline{\lambda}_{f_k}}{\rho_{f_k}} \le \liminf_{r\to\infty} \frac{\log^{[n+1]} T(r, F_n^1)}{\log\ T(r, f_k^{(l)})} \le \limsup_{r\to\infty} \frac{\log^{[n+1]} T(r, F_n^1)}{\log\ T(r, f_k^{(l)})} \le \frac{\overline{\rho_f}_k}{\lambda_{f_k}}$$

when $n = km$, $m \in \mathbb{N}$, *where* $f^{(l)}$ *denote the* l*-th derivative of* f.

Proof. When $n = km$, $m \in \mathbb{N}$, then for given $\varepsilon (0 < \varepsilon < min\{\lambda_{f_1}, \lambda_{f_2}, ..., \lambda_{f_k}\})$ we get from Lemma 2.6 for all large values of r

$$\begin{aligned} \log^{[n-1]}\ T(r, F_n^1) &\ge (\lambda_{f_{k-1}} - \varepsilon) \log M\left(\frac{r}{4^{n-1}}, f_k\right) + O(1) \\ &\ge (\lambda_{f_{k-1}} - \varepsilon) T\left(\frac{r}{4^{n-1}}, f_k\right) + O(1). \\ \text{So,} \quad \log^{[n+1]}\ T(r, F_n^1) &\ge \log^{[2]} T\left(\frac{r}{4^{n-1}}, f_k\right) + O(1). \end{aligned}$$

So for all large values of r

$$\frac{\log^{[n+1]} T(r, F_n^1)}{\log\ T(r, f_k^{(l)})} \ge \frac{\log^{[2]} T\left(\frac{r}{4^{n-1}}, f_k\right)}{\log\ \frac{r}{4^{n-1}}} . \frac{\log \frac{r}{4^{n-1}}}{\log\ T(r, f_k^{(l)})} + o(1). \tag{8}$$

Again since

$$\limsup_{r\to\infty} \frac{\log T(r, f_k^{(l)})}{\log r} = \rho_{f_k},$$

so for all large values of r and arbitrary $\varepsilon > 0$ we get

$$\log T(r, f_k^{(l)}) < (\rho_{f_k} + \varepsilon) \log r. \tag{9}$$

Since $\varepsilon > 0$ is arbitrary, so from (8) and (9) we get

$$\begin{aligned} \frac{\log^{[n+1]} T(r, F_n^1)}{\log\ T(r, f_k^{(l)})} &\ge \frac{\log^{[2]} T\left(\frac{r}{4^{n-1}}, f_k\right)}{\log\ \frac{r}{4^{n-1}}} . \left(\frac{\log r - \log 4^{n-1}}{(\rho_{f_k} + \varepsilon) \log\ r}\right) \\ \text{So,} \quad \liminf_{r\to\infty} \frac{\log^{[n+1]} T(r, F_n^1)}{\log\ T(r, f_k^{(l)})} &\ge \frac{\overline{\lambda}_{f_k}}{\rho_{f_k}}. \end{aligned} \tag{10}$$

Again from Lemma 2.6 for all large values of r

$$\begin{aligned} \log^{[n-1]}\ T(r, F_n^1) &\le (\rho_{f_{k-1}} + \varepsilon) \log M(r, f_k) + O(1) \\ \text{i.e.} \quad \frac{\log^{[n+1]} T(r, F_n^1)}{\log\ T(r, f_k^{(l)})} &\le \frac{\log^{[3]} M(r, f_k)}{\log\ T(r, f_k^{(l)})} + o(1). \end{aligned} \tag{11}$$

Since

$$\lim_{r\to\infty}\inf \frac{\log T(r, f_k^{(l)})}{\log r} = \lambda_{f_k},$$

so for all large values of r and arbitrary $\varepsilon(0 < \varepsilon < \lambda_{f_k})$ we have

$$\log T(r, f_k^{(l)}) > (\lambda_{f_k} - \varepsilon)\log r. \tag{12}$$

Since $\varepsilon > 0$ is arbitrary, so from (11) and (12) we have

$$\lim_{r\to\infty}\sup \frac{\log^{[n+1]} T(r, F_n^1)}{\log\ T(r, f_k^{(l)})} \le \frac{\overline{\rho_{f_k}}}{\lambda_{f_k}}. \tag{13}$$

From (10) and (13) we get the result. This completes the proof.

Corollary 3.6. *Let $f_1, f_2, \ldots, f_k$ be k entire functions such that $0 < \lambda_{f_i} \le \rho_{f_i} < \infty, i = 1, 2, \ldots, k$. Then for $l = 0, 1, 2, 3, \ldots$*

$$\frac{\overline{\lambda}_{f_1}}{\rho_{f_1}} \le \lim_{r\to\infty}\inf \frac{\log^{[n+1]} T(r, F_n^1)}{\log\ T(r, f_1^{(l)})} \le \lim_{r\to\infty}\sup \frac{\log^{[n+1]} T(r, F_n^1)}{\log\ T(r, f_1^{(l)})} \le \frac{\overline{\rho}_{f_1}}{\lambda_{f_1}}$$

when $n = km - (k-1)$, $m \in \mathbb{N}$ where $f^{(l)}$ denote the l-th derivative of f.

Theorem 3.7. *Let $f_1, f_2, \ldots, f_k$ be k entire functions such that $0 < \lambda_{f_i} \le \rho_{f_i} < \infty, i = 1, 2, \ldots, k$. Then*

$$\frac{\lambda_{f_k}}{\rho_{f_k}} \le \lim_{r\to\infty}\inf \frac{\log^{[n]} T(r, F_n^1)}{\log\ T(r, f_k)} \le 1 \le \lim_{r\to\infty}\sup \frac{\log^{[n]} T(r, F_n^1)}{\log\ T(r, f_k)} \le \frac{\rho_{f_k}}{\lambda_{f_k}}$$

when $n = km$, $m \in \mathbb{N}$.

Proof. When $n = km$, $m \in \mathbb{N}$. Then for given $\varepsilon(0 < \varepsilon < min\{\lambda_{f_1}, \lambda_{f_2}, \ldots, \lambda_{f_k}\})$ from Lemma 2.6 and Lemma 2.3 for all large values of r we have

$$\begin{aligned} \log^{[n-1]}\ T(r, F_n^1) &\le (\rho_{f_{k-1}} + \varepsilon)\log M(r, f_k) + O(1) \\ \text{i.e.}\quad \log^{[n]}\ T(r, F_n^1) &\le \log^{[2]} M(r, f_k) + O(1) \qquad (14) \\ \text{i.e.}\quad \frac{\log^{[n]} T(r, F_n^1)}{\log\ T(r, f_k)} &\le \frac{\log^{[2]} M(r, f_k)}{\log\ T(r, f_k)} + o(1) \\ \text{i.e.}\ \lim_{r\to\infty}\inf \frac{\log^{[n]} T(r, F_n^1)}{\log\ T(r, f_k)} &\le 1. \qquad (15) \end{aligned}$$

Again since

$$\log^{[n-1]}\ T(r,F_n^1) \geq (\lambda_{f_{k-1}}-\varepsilon)\log M\left(\frac{r}{4^{n-1}},f_k\right)+O(1)$$

so,

$$\begin{aligned}\frac{\log^{[n]} T(r,F_n^1)}{\log\ T(r,f_k)} &\geq \frac{\log T\left(\frac{r}{4^{n-1}},f_k\right)}{\log\ \frac{r}{4^{n-1}}}.\left(\frac{\log r-\log 4^{n-1}}{(\rho_{f_k}+\varepsilon)\log\ r}\right)+o(1)\\ \text{i.e.}\quad \liminf_{r\to\infty}\frac{\log^{[n]} T(r,F_n^1)}{\log\ T(r,f_k)} &\geq \frac{\lambda_{f_k}}{\rho_{f_k}}. \qquad (16)\end{aligned}$$

From (14), we get for all large values of r

$$\begin{aligned}\frac{\log^{[n]} T(r,F_n^1)}{\log\ T(r,f_k)} &\leq \frac{\log^{[2]} M(r,f_k)}{\log r}\frac{\log r}{\log T(r,f_k)}+o(1)\\ \therefore \limsup_{r\to\infty}\frac{\log^{[n]} T(r,F_n^1)}{\log T(r,f_k)} &\leq \frac{\rho_{f_k}}{\lambda_{f_k}}. \qquad (17)\end{aligned}$$

Also from Lemma 2.6,

$$\begin{aligned}\log^{[n-1]}\ T(r,F_n^1) &\geq (\lambda_{f_{k-1}}-\varepsilon)\log M\left(\frac{r}{4^{n-1}},f_k\right)+O(1)\\ \text{i.e.}\quad \log^{[n]}\ T(r,F_n^1) &\geq \log^{[2]} M\left(\frac{r}{4^{n-1}},f_k\right)+O(1). \qquad (18)\end{aligned}$$

We have from (5) for all large values of r and for $\delta>0$ and $\varepsilon(0<\varepsilon<1)$

$$\begin{aligned}\log M\left(\frac{r}{4^{n-1}},f_k\right) &> (1-\varepsilon)\frac{\left(\frac{r}{4^{n-1}}\right)^{\lambda_{f_k}+\delta}}{\left(\frac{r}{4^{n-1}}\right)^{\lambda_{f_k}+\delta-\lambda_{f_k}\left(\frac{r}{4^{n-1}}\right)}}\\ &\geq \frac{1-\varepsilon}{(4^{n-1})^{\lambda_{f_k}+\delta}}r^{\lambda_{f_k}(r)}\end{aligned}$$

since $r^{\lambda_{f_k}+\delta-\lambda_{f_k}(r)}$ is an increasing function of r.

Therefore by (4) we get for a sequence of value of $r\to\infty$

$$\begin{aligned}\log M\left(\frac{r}{4^{n-1}},f_k\right) &\geq \frac{1-\varepsilon}{1+\varepsilon}\frac{1}{(4^{n-1})^{\lambda_{f_k}+\delta}}T(r,f_k)\\ \text{i.e.}\ \log^{[2]} M\left(\frac{r}{4^{n-1}},f_k\right) &\geq \log T(r,f_k)+O(1). \qquad (19)\end{aligned}$$

Now from (18) and (19)

$$\limsup_{r\to\infty} \frac{\log^{[n]} T(r, F_n^1)}{\log\ T(r, f_k)} \geq 1. \tag{20}$$

So from (15), (16), (17) and (20) we get the result when $n = km$, $m \in \mathbb{N}$.
This completes the proof.

Corollary 3.8. *Let $f_1, f_2, ..., f_k$ be k entire functions such that $0 < \lambda_{f_i} \leq \rho_{f_i} < \infty, i = 1, 2, ..., k$. Then*

$$\frac{\lambda_{f_1}}{\rho_{f_1}} \leq \liminf_{r\to\infty} \frac{\log^{[n]} T(r, F_n^1)}{\log\ T(r, f_1)} \leq 1 \leq \limsup_{r\to\infty} \frac{\log^{[n]} T(r, F_n^1)}{\log\ T(r, f_1)} \leq \frac{\rho_{f_1}}{\lambda_{f_1}}$$

when $n = km - (k-1)$, $m \in \mathbb{N}$.

Theorem 3.9. *Let $f_1, f_2, ..., f_k$ be k entire functions such that $0 < \lambda_{f_i} \leq \rho_{f_i} < \infty, i = 1, 2, ..., k$. Then for $l = 0, 1, 2, 3, ...$*

$$\frac{\lambda_{f_k}}{\rho_{f_k}} \leq \liminf_{r\to\infty} \frac{\log^{[n]} T(r, F_n^1)}{\log\ T(r, f_k^{(l)})} \leq \limsup_{r\to\infty} \frac{\log^{[n]} T(r, F_n^1)}{\log\ T(r, f_k^{(l)})} \leq \frac{\rho_{f_k}}{\lambda_{f_k}}$$

when $n = km$, $m \in \mathbb{N}$.

Proof. Let $n = km$. Then for given $\varepsilon (0 < \varepsilon < min\{\lambda_{f_1}, \lambda_{f_2}, ..., \lambda_{f_k}\})$ we have from Lemma 2.6 for all large values of r

$$\begin{aligned} \log^{[n-1]}\ T(r, F_n^1) &\leq (\rho_{f_{k-1}} + \varepsilon) \log M(r, f_k) + O(1) \\ \text{i.e.} \quad \log^{[n]}\ T(r, F_n^1) &\leq \log^{[2]} M(r, f_k) + O(1). \end{aligned}$$

We know that

$$\liminf_{r\to\infty} \frac{\log T(r, f_k^{(l)})}{\log r} = \lambda_{f_k}.$$

Now

$$\begin{aligned} \limsup_{r\to\infty} \frac{\log^{[n]} T(r, F_n^1)}{\log\ T(r, f_k^{(l)})} &\leq \limsup_{r\to\infty} \frac{\log^{[2]} M(r, f_k)}{\log\ T(r, f_k^{(l)})} \\ &\leq \limsup_{r\to\infty} \left[\frac{\log^{[2]} M(r, f_k)}{\log r} . \frac{\log r}{\log T(r, f_k^{(l)})} \right] \\ &= \frac{\rho_{f_k}}{\lambda_{f_k}}. \end{aligned} \tag{21}$$

Again from Lemma 2.6, we have for all large values of r

$$\begin{aligned}\log^{[n-1]} T(r,F_n^1) &\geq (\lambda_{f_{k-1}}-\varepsilon)\log M\left(\frac{r}{4^{n-1}},f_k\right)+O(1)\\ &\geq (\lambda_{f_{k-1}}-\varepsilon)\left(\frac{r}{4^{n-1}}\right)^{\lambda_{f_k}-\varepsilon}+O(1)\\ \text{i.e.,}\quad \log^{[n]} T(r,F_n^1) &\geq (\lambda_{f_k}-\varepsilon)\log r+O(1).\end{aligned}$$

Also

$$\log T(r,f_k^{(l)})<(\rho_{f_k}+\varepsilon)\log r.$$

Therefore,

$$\frac{\log^{[n]} T(r,F_n^1)}{\log T(r,f_k^{(l)})}\geq\frac{(\lambda_{f_k}-\varepsilon)\log r+O(1)}{(\rho_{f_k}+\varepsilon)\log r}.$$

Since $\varepsilon>0$ is arbitrary we get

$$\liminf_{r\to\infty}\frac{\log^{[n]} T(r,F_n^1)}{\log T(r,f_k^{(l)})}\geq\frac{\lambda_{f_k}}{\rho_{f_k}}. \tag{22}$$

Therefore from (21) and (22) we have the result for $n=km$, $m\in\mathbb{N}$.
This proves the Theorem.

Corollary 3.10. *Let $f_1,f_2,...,f_k$ be k entire functions such that $0<\lambda_{f_i}\leq\rho_{f_i}<\infty, i=1,2,...,k$. Then for $l=0,1,2,3,.....$*

$$\frac{\lambda_{f_1}}{\rho_{f_1}}\leq\liminf_{r\to\infty}\frac{\log^{[n]} T(r,F_n^1)}{\log T(r,f_1^{(l)})}\leq\limsup_{r\to\infty}\frac{\log^{[n]} T(r,F_n^1)}{\log T(r,f_1^{(l)})}\leq\frac{\rho_{f_1}}{\lambda_{f_1}}$$

when $n=km-(k-1)$, $m\in\mathbb{N}$.

Theorem 3.11. *Let $f_1,f_2,...,f_k$ be k entire functions such that $0<\lambda_{f_1}\leq\rho_{f_1}<\infty$ and $\rho_{f_2},\rho_{f_3},...,\rho_{f_k}<\infty$. Then for $l=0,1,2,3,......$*

$$\lim_{r\to\infty}\frac{\log^{[n-1]} T(r,F_n^1)}{T(\exp(r),f_1^{(l)})}=0\quad\textit{for all natural number } n(\geq k).$$

Proof. Suppose $n=km$, $m\in\mathbb{N}$. Then by Lemma 2.6 for all sufficiently large values of r and $\varepsilon(0<\varepsilon<\lambda_{f_1})$

$$\begin{aligned}\log^{[n-1]} T(r,F_n^1) &\leq (\rho_{f_{k-1}}+\varepsilon)\log M(r,f_k)+O(1),\ \log M(r,f_k)<r^{\rho_{f_k}+\varepsilon}\\ \text{and}\quad T(\exp(r),f_1^{(l)}) &> e^{r^{(\lambda_{f_1}-\varepsilon)}}.\end{aligned}$$

So

$$\frac{\log^{[n-1]} T(r, F_n^1)}{T(\exp(r), f_1^{(l)})} \leq \frac{(\rho_{f_{k-1}} + \varepsilon) r^{\rho_{f_k} + \varepsilon}}{e^{r^{(\lambda_{f_1} - \varepsilon)}}} + o(1)$$

$$\therefore \lim_{r\to\infty} \frac{\log^{[n-1]} T(r, F_n^1)}{T(\exp(r), f_1^{(l)})} = 0.$$

When $n = km - 1$, $m \in \mathbb{N}$. Then since

$$\log^{[n-1]} T(r, F_n^1) \leq (\rho_{f_{k-2}} + \varepsilon) \log M(r, f_{k-1}) + O(1)$$
$$\text{and} \quad \log M(r, f_{k-1}) < r^{\rho_{f_{k-1}} + \varepsilon}$$

so

$$\lim_{r\to\infty} \frac{\log^{[n-1]} T(r, F_n^1)}{T(\exp(r), f_1^{(l)})} = 0.$$

Similarly for $n = km - (k-1)$, $m \in \mathbb{N}$.

Remark 3.12. *The condition $\rho_{f_2}, \rho_{f_3}, ..., \rho_{f_k} < \infty$ is the necessary for Theorem 3.11, which is shown by the following example.*

Example 3.13. Let $f_1 = \exp z$, $f_2 = \exp^{[2]} z$, $f_3(z) = \exp^{[3]} z, ..., f_k(z) = \exp^{[k]}(z)$. Then $\lambda_{f_1} = \rho_{f_1} = 1$ and $\rho_{f_2} = \rho_{f_3} = ... = \rho_{f_k} = \infty$.

Now when $n = km$, $m \in \mathbb{N}$

$$F_n^1 = \exp^{[\frac{(k+1)}{2} n]} z.$$

Therefore,

$$3T(2r, F_n^1) \geq \log M(r, F_n^1) = \exp^{[\frac{(k+1)}{2} n - 1]} r$$
$$\text{i.e.} \quad T(r, F_n^1) \geq \frac{1}{3} \exp^{[\frac{(k+1)}{2} n - 1]} \frac{r}{2}$$
$$\text{So,} \quad \log^{[n-1]} T(r, F_n^1) \geq \exp^{[\frac{(k-1)}{2} n]} \frac{r}{2} + o(1).$$

Also

$$T(\exp(r), f_1^{(l)}) = \frac{e^r}{\pi}.$$

Therefore when $n = km$, $m \in \mathbb{N}$

$$\frac{\log^{[n-1]} T(r, F_n^1)}{T(\exp(r), f_1^{(l)})} \geq \frac{\exp^{[\frac{(k-1)}{2} n]} \frac{r}{2} + o(1)}{e^r / \pi} \to \infty \quad \text{as} \quad r \to \infty.$$

Similarly when $n = km - 1, km - 2, ..., km - (k - 1)$, $m \in \mathbb{N}$.

Corollary 3.14. *Let $f_1, f_2, ..., f_k$ be k entire functions such that $0 < \lambda_{f_k} \leq \rho_{f_k} < \infty$ and $\rho_{f_1}, \rho_{f_2}, ..., \rho_{f_{k-1}} < \infty$. Then for $l = 0, 1, 2, 3, ...$ and for $n = km - (k - 1)$, $m \in \mathbb{N}$*

$$\lim_{r\to\infty} \frac{\log^{[n-1]} T(r, F_n^1)}{T(\exp(r), f_k^{(l)})} = 0 \quad \textit{for all natural number } n(\geq k).$$

Theorem 3.15. *Let $f_1, f_2, ..., f_k$ be k entire functions such that $\rho_{f_k} < \lambda_{f_{k-1}} \leq \rho_{f_{k-1}} < \infty$. Then for $k = 0, 1, 2, 3, ...$*

$$\lim_{r\to\infty} \frac{\log^{[n-1]} T(r, F_n^1)}{T(r, f_{k-1}^{(l)})} = 0 \quad \textit{when } n = km, \ m \in \mathbb{N}.$$

Proof. We have from Lemma 2.6 for $\varepsilon(> 0)$ with $\rho_{f_k} + \varepsilon < \lambda_{f_{k-1}} - \varepsilon$ and for large values of r

$$\begin{aligned} \log^{[n-1]} \ T(r, F_n^1) &\leq (\rho_{f_{k-1}} + \varepsilon) \log M(r, f_k) + O(1), \log M(r, f_k) < r^{\rho_{f_k}+\varepsilon} \\ \text{and} \ \ T(r, f_{k-1}^{(l)}) &> r^{\lambda_{f_{k-1}}-\varepsilon}. \end{aligned}$$

So

$$\begin{aligned} \frac{\log^{[n-1]} T(r, F_n^1)}{T(r, f_{k-1}^{(l)})} &\leq \frac{(\rho_{f_{k-1}} + \varepsilon) r^{\rho_{f_k}+\varepsilon}}{r^{\lambda_{f_{k-1}}-\varepsilon}} + o(1) \\ \text{i.e.} \quad \lim_{r\to\infty} \frac{\log^{[n-1]} T(r, F_n^1)}{T(r, f_{k-1}^{(l)})} &= 0. \end{aligned}$$

This completes the proof.

Corollary 3.16. *Let $f_1, f_2, ..., f_k$ be k entire functions such that $\rho_{f_1} < \lambda_{f_k} \leq \rho_{f_k} < \infty$. Then for $k = 0, 1, 2, 3, ...$,*

$$\lim_{r\to\infty} \frac{\log^{[n-1]} T(r, F_n^1)}{T(r, f_k^{(l)})} = 0 \quad \textit{when } n = km - (k - 1), m \in \mathbb{N}.$$

In [7] Lahiri and Datta proved the following theorem.

Theorem A. *Let f and g be two transcendental entire functions such that*

$$\textit{(i)} \ 0 < \lambda_g \leq \rho_g < \infty, \quad \textit{(ii)} \ \lambda_f > 0, \quad \textit{and} \quad \textit{(iii)} \ \ \delta(0; f) < 1.$$

Then for any real number A

$$\limsup_{r\to\infty} \frac{\log T(r, f_o g)}{\log \ T(r^A, g^{(l)})} = \infty$$

for $l = 0, 1, 2, 3, \ldots$.

Theorem 3.17. *Let* $f_1, f_2, \ldots, f_k$ *be k transcendental entire functions such that*

$$\text{(i) } 0 < \lambda_{f_k} \le \rho_{f_k} < \infty, \quad \text{(ii) } \lambda_{f_{k-1}} > 0, \quad \text{and} \quad \text{(iii)} \quad \delta(0; f_{k-1}) < 1.$$

Then for any real number A

$$\limsup_{r\to\infty} \frac{\log^{[n-1]} T(r, F_n^1)}{\log \ T(r^A, f_k^{(l)})} = \infty$$

for $l = 0, 1, 2, 3, \ldots\ldots$ *and* $n = km, m \in \mathbb{N}$.

Proof. When $n = km, m \in \mathbb{N}$ then from (2)

$$\begin{aligned} \log^{[n-2]} \ T(r, F_n^1) &\ge (\lambda_{f_{k-2}} - \varepsilon) \log M\left(\frac{r}{4^{n-2}}, f_{k-1} f_k\right) + O(1) \\ &\ge (\lambda_{f_{k-2}} - \varepsilon) T\left(\frac{r}{4^{n-2}}, f_{k-1} f_k\right) + O(1) \\ \text{i.e,} \quad \log^{[n-1]} \ T(r, F_n^1) &\ge \log T\left(\frac{r}{4^{n-2}}, f_{k-1} f_k\right) + O(1) \ . \end{aligned}$$

So

$$\limsup_{r\to\infty} \frac{\log^{[n-1]} T(r, F_n^1)}{\log \ T(r^A, f_k^{(l)})} \ge \limsup_{r\to\infty} \frac{\log T\left(\frac{r}{4^{n-2}}, f_{k-1} f_k\right)}{\log \ T(r^A, f_k^{(l)})}$$

$$\ge \limsup_{r\to\infty} \left[\frac{\log T\left(\frac{r}{4^{n-2}}, f_{k-1} f_k\right)}{\log \ T\left(\left(\frac{r}{4^{n-2}}\right)^A, f_k^{(l)}\right)} \frac{\log \ T\left(\left(\frac{r}{4^{n-2}}\right)^A, f_k^{(l)}\right)}{\log \ T(r^A, f_k^{(l)})} \right]. \tag{23}$$

Again

$$\limsup_{r\to\infty} \frac{\log \ T\left(\left(\frac{r}{4^{n-2}}\right)^A, f_k^{(l)}\right)}{\log \ T(r^A, f_k^{(l)})} \ge \frac{\lambda_{f_k}}{\rho_{f_k}}. \tag{24}$$

Therefore from Theorem A we have the result by using (23) and (24). This completes the proof.

Corollary 3.18. *Let* $f_1, f_2, \ldots, f_k$ *be k transcendental entire functions such that*

$$\text{(i) } 0 < \lambda_{f_1} \le \rho_{f_1} < \infty, \quad \text{(ii) } \lambda_{f_k} > 0, \quad \text{and} \quad \text{(iii)} \quad \delta(0; f_k) < 1.$$

Then for any real number A

$$\limsup_{r\to\infty} \frac{\log^{[n-1]} T(r, F_n^1)}{\log\ T(r^A, f_1^{(l)})} = \infty$$

for $l = 0, 1, 2, 3, ...$ *and* $n = km - (k-1), m \in \mathbb{N}$.

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