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EULERIAN OF THE ZERO DIVISOR GRAPH $\Gamma[\mathbb{Z}_n]$

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Abstract: The Zero divisor Graph of a commutative ring R, denoted by $\Gamma[R]$, is a graph whose vertices are non-zero zero divisors of R and two vertices are adjacent if their product is zero. We consider the zero divisor graph $\Gamma[\mathbb{Z}_n]$, for any natural number n and find out which graphs are Eulerian graphs.

Keywords and Phrases: Zero divisor graph, Euler tour, Euler graph.

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1. Introduction

The concept of the Zero divisor graph of a ring R was first introduced by I. Beck [3] in 1988 and later on Anderson and Livingston [2], Akbari and Mohammadian [1] continued the study of zero divisor graph by considering only the non-zero zero divisors. The concepts of the Euler graph found in [4]. In this paper we introduce the concepts of the Euler graph to the zero divisor graph $\Gamma[\mathbb{Z}_n]$ and identify which zero divisors graphs are Eulerian.

In this article, section 2, is about the preliminaries and notations related to zero divisor graph of a commutative ring R, in section 3, we derive the Euler graphs of a zero divisor graph $\Gamma[\mathbb{Z}_{p^n}]$, and in section 4, we discuss about Euler graphs of $\Gamma[\mathbb{Z}_n]$ for any natural number n.

2. Preliminaries and Notations

Definition 2.1. Zero divisor Graph [1,2],

Let R be a commutative ring with unity and $Z[R]$ be the set of its zero divisors. Then the zero divisor graph of R denoted by $\Gamma[R]$, is the graph(undirected) with vertex set $Z^*[R] = Z[R] - \{\mathbf{0}\}$, the non-zero zero divisors of R, such that two vertices $v, w \in Z^*[R]$ are adjacent if $vw = 0$.

Definition 2.2. Euler tour [4],

An Euler tour of a graph G is a tour which includes each edge of the graph G exactly once.

Definition 2.3. Euler tour [4],

A graph G is called Euler graph or Eulerain if it has an Euler tour.

Theorem 2.4. [4] *A connected graph is Euler iff the degree of every vertex is even.*

3. Eulerian of The zero divisor graph $\Gamma[\mathbb{Z}_{p^n}]$

In this section, we discuss the Eulerian of the zero divisor graph $\Gamma[\mathbb{Z}_{p^n}]$ where p is a prime number.
To start with, we consider the zero divisor graph $\Gamma[\mathbb{Z}_n]$ for $n = p^2$.

Theorem 3.1. *The zero divisor graph $\Gamma[\mathbb{Z}_{p^2}]$ is a Euler graph if and only if $p > 2$.*

Proof. Consider zero divisor graph $\Gamma[\mathbb{Z}_{p^2}]$.
The vertex set is $A = \{kp \mid k = 1, 2, 3, ..., p-1\}$ and so $|A| = (p-1)$.
As product of any two vertices is zero, they are adjacent and so the corresponding graph is a complete graph on (p-1) vertices that is, $\Gamma[\mathbb{Z}_{p^2}] = K_{p-1}$.
As the graph is complete, the degree of each and every vertex of it is (p-1).
If $p > 2$ then every prime greater than 2 is odd and hence the degree of each vertex is even.Thus $\Gamma[\mathbb{Z}_{p^2}]$ is Eulerian.
For $p = 2$ then the corresponding graph has no Euler path as it consists of only one vertex, thus $\Gamma[\mathbb{Z}_4]$ is not Eulerian.

Theorem 3.2. *The zero divisor graph $\Gamma[\mathbb{Z}_{p^3}]$ is not an Euler graph, for any prime p.*

Proof. Consider the zero divisor graph $\Gamma[\mathbb{Z}_{p^3}]$.
Here, we divide the elements(vertices) of $\Gamma[\mathbb{Z}_{p^3}]$ into two disjoint sets namely mul-

tiples of p and the multiples of p^2 which are given by

$$A = \{kp \mid k = 1, 2, 3, ..., p^2 - 1 \text{ and } k \nmid p\}$$
$$B = \{lp^2 \mid l = 1, 2, 3, ..., p - 1\}$$

with cardinality $|A| = p(p-1)$ and $|B| = (p-1)$.
As every element of A is adjacent only with the elements of B, the degree of each and every vertex of A is $(p-1)$ which is even.
Also every element of B is adjacent with itself and with every element of A.
Therefore the degree of each and every vertex of B is given by $|A| + |B| - 1$ that is $(p^2 - 2)$ which is odd.
Hence $\Gamma[\mathbb{Z}_{p^3}]$ is not Eulerian.
If $p = 2$, then degree of each vertex of A is $p - 1$ which is odd. Therefore the zero divisor graph $\Gamma[\mathbb{Z}_{p^3}]$ is not an Euler graph.
With similar arguments, we prove the more general case in the following theorem.

Theorem 3.3. *The zero divisor graph* $\Gamma[\mathbb{Z}_{p^n}]$ *is not Eulerian, for any prime* p.

Proof. We divide the elements(vertices) of $\Gamma[\mathbb{Z}_{p^n}]$ into $n-1$ disjoint sets namely multiples of p, multiples of p^2... multiples of p^{n-1}, given by

$$A_1 = \{k_1 p \mid k_1 = 1, 2, 3, ..., p^{n-1} - 1 \text{ and } k_1 \nmid p\}$$
$$A_2 = \{k_2 p^2 \mid k_2 = 1, 2, 3, ..., p^{n-2} - 1 \text{ and } k_2 \nmid p^2\}$$
$$A_i = \{k_i p^i \mid k_i = 1, 2, 3, ..., p^{n-i} - 1 \text{ and } k_i \nmid p^i\}$$

with cardinality $|A_i| = (p^{n-i} - p^{n-i-1})$, for $i = 1, 2, ...n - 1$.
Also the smallest set is A_{n-1} of order $p - 1$.
Now the degree of an element v_i in A_i is $p^i - 2$ which is odd $\forall\ i = [\frac{n}{2}]$ a greatest integer part function, since the elements of A_i are adjacent with itself and also with A_j for $j \geq [\frac{n}{2}]$.
We can make a similar argument for all other sets i.e., every element of A_i is adjacent with every element of A_{n-j} where $j \leq i$, therefore the degree of every vertex of A_i is $\sum_{j=1}^{i}(p^{n-j} - p^{n-j-1}) - 1 = (p^{n-1} - 2)$ which is odd.
Hence the zero divisor graph $\Gamma[\mathbb{Z}_{p^n}]$ is not Eulerian.
If $p = 2$, then the degree of an element v_1 in A_1 is $p - 1$, which is odd. Therefore, for any prime, the zero divisor graph $\Gamma[\mathbb{Z}_{p^n}]$ is not Eulerian.

4. Eulerian of the zero divisor graph $\Gamma[\mathbb{Z}_n]$

In this section we discuss the Eulerian of the zero divisor graph $\Gamma[\mathbb{Z}_n]$ where $n = p_1^{\alpha_1} p_2^{\alpha_2} .....p_k^{\alpha_k}$.
To start with, we consider $n = pq$.

Theorem 4.1. *The zero divisor graph $\Gamma[\mathbb{Z}_{pq}]$ is a Euler graph iff p and q are odd.*

Proof. Consider the zero divisor graph $\Gamma[\mathbb{Z}_{pq}]$.
clearly $\Gamma[\mathbb{Z}_{pq}]$ is a complete bipartite graph, the vertex sets are given by

$$A = \{kp \,|\, k = 1, 2, 3, ..., p-1 \,\text{and}\, k \nmid q\}$$
$$B = \{lq \,|\, l = 1, 2, 3, ..., q-1 \,\text{and}\, k \nmid p\}$$

with cardinality $|A| = (p-1)$ and $|B| = (q-1)$.
If p and q are odd, then $(p-1)$ and $(q-1)$ are even implies the degree of every vertex of the graph is even and thus the respective graph is an Euler graph.
If $p \,\text{or}\, q = 2$, then clearly the graph is not Eulerian as the degree of the atleast one vertex is odd.

Theorem 4.2. *The zero divisor graph $\Gamma[\mathbb{Z}_{p^\alpha q^\beta}]$ is not Eulerian for all $\alpha, \beta \neq 1$.*
Proof. Consider the zero divisor graph $\Gamma[\mathbb{Z}_{p^\alpha q^\beta}]$.
Here, we divide the vertices of $\Gamma[\mathbb{Z}_{p^\alpha q^\beta}]$ into disjoint sets namely multiples of p^i, multiples of q^j and multiples of $p^i q^j$ given by

$$A_{p^i} = \{r_i p^i \,|\, r_i = 1, 2, 3, ..., p^i - 1 \,\text{and}\, r_i \nmid p^i\}$$
$$A_{q^j} = \{s_j q^j \,|\, s_j = 1, 2, 3, ..., q^j - 1 \,\text{and}\, s_j \nmid q^j\}$$
$$A_{p^i q^j} = \{t_{ij} p^i q^j \,|\, t_{ij} = 1, 2, 3, ..., p^i q^j - 1 \,\text{and}\, t_{ij} \nmid p^i \,\text{and}\, t_{ij} \nmid q^j\}.$$

Then the order of the sets are $|A_{p^i}| = (p^i - 1)$, $|A_{q^j}| = (q^j - 1)$ and $|A_{p^i q^j}| = (p^i - 1)(q^j - 1)$.
Assume that both p and q are odd primes.
Since every element of the set A_{p^i} is adjacent with the elements of A_{p^j}, the degree of each and every vertex of the setA_{p^i} is $(q^j - 1)$.
Similarly the degree of each and every vertex of the set A_{p^j} is $(p^i - 1)$ and the degree of each and every vertex of the set $A_{p^i q^j}$ is $|A_{p^i}| + A_{p^j}| + |A_{p^i q^j}| - 1 = (p^i - 1) + (q^j - 1) + (p^i - 1)(q^j - 1) = (p^i q^j - 2)$, which is odd.
Thus the degree of the vertices of the corresponding sets is odd.
Hence the zero divisor graph $\Gamma[\mathbb{Z}_{p^\alpha q^\beta}]$ is not Eulerian.
If one of $p \,\text{or}\, q = 2$, then also the graph is not Eulerian as the degree of the atleast one vertex is $(p^i - 1)$ or $(q^j - 1)$ which is odd. Also the degree of the elements of A_p is $(p-1)$ as the elements of these set are adjacent only with the elements of the set $A_{p^{\alpha-1} q^\beta}$.

Theorem 4.3. *The zero divisor graph $\Gamma[\mathbb{Z}_n]$ where $n = p_1^{\alpha_1} p_2^{\alpha_2} ... p_k^{\alpha_k}$ is not Eulerian for $\alpha_i \geq 2$ where $i = 1, 2, ..., k$.*

Proof. Consider a zero divisor graph $\Gamma[\mathbb{Z}_n]$ where $n = p_1^{\alpha_1} p_2^{\alpha_2} .....p_k^{\alpha_k}$.
Here, we divide the elements(vertices) of $\Gamma[\mathbb{Z}_n]$ into the corresponding disjoint sets of product of all possible powers of given primes like set of powers of p_j^i, set of product of powers of $p_i^r p_j^s$ and so on.
Among these sets, we consider the sets of the form
$A_i = \{m(p_1^{\alpha_1} p_2^{\alpha_2} ...p_i^{\alpha_{i-1}} p_i^{\alpha_{i+1}} ....p_k^{\alpha_k})\}$ with $|A_i| = p_i^{\alpha_i} - 1$.
Now consider the set $A_{p_i^{\alpha}{}_i} = \{tp_i^{\alpha_i} \mid t \nmid p_i^{\alpha_i}\}$.
Assume that all the primes are odd.
Since the elements of $A_{p_j^i}$ are adjacent only with the vertices of A_j, the degree of each and every vertex of the set $A_{p_j^i}$ is $p_i^{\alpha_i} - 1$ which is odd. Hence the zero divisor graph $\Gamma[\mathbb{Z}_n]$ where $n = p_1^{\alpha_1} p_2^{\alpha_2} ...p_k^{\alpha_k}$ is not Eulerian for $\alpha_i \geq 2$ where $i = 1, 2, ..., k$.
If one of $p_i = 2$, then also the graph is not Eulerian as the degree of the atleast one vertex is $p_i^{\alpha_i} - 1$, which is odd.

5. Conclusion

We conclude that the zero divisor graph $\Gamma[\mathbb{Z}_n]$ is Eulerain if and only if either $n = p^2$ or $n = pq$ where p and q are distinct primes. Otherwise not a Eulerian.

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