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Certain unified integral involving Generalized Bessel-Maitland function

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Abstract: MacRobert in his research paper established certain new finite integral formula, which is expressed in terms of gamma functions. Using the result of MacRobert, in this paper, we present a new integral formula involving the generalized Bessel-Maitland function, which is expressed in terms of generalized (Wright) hypergeometric function. Some interesting special cases of our main result are also considered.
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1. Introduction

Many unified integrals involving special functions of mathematical physics have been presented by a number of authors, for example, Rathie ([8], [9]), Ali [1], Choi and Agarwal [3] and Choi et al. [4]. Motivated by the above-mentioned works, in the present paper, we establish a new unified integral formula involving the Bessel-Maitland function. Some special cases of our main result are also considered.
The Bessel function $J_\nu(z)$ of the first kind (and order ν), defined by (See [7])

$$J_\nu(z) = \sum_{m=0}^{\infty} \frac{(-1)^m \ (z/2)^{\nu+2m}}{m! \ \Gamma(\nu+m+1)}, \tag{1.1}$$

it is well known that

$$J_{-\frac{1}{2}}(z) = \sqrt{\frac{2}{\pi z}} \ \cos z \tag{1.2}$$

and

$$J_{\frac{1}{2}}(z) = \sqrt{\frac{2}{\pi z}}\ \sin z. \tag{1.3}$$

An interesting generalization of the (classical) Bessel function $J_\nu(z)$ due to Wright [13], who studied the function $J_\nu^\mu(z)$ defined by

$$J_\nu^\mu(z) = \sum_{m=0}^{\infty} \frac{(-z)^m}{m!\ \Gamma(\nu+\mu m+1)} \qquad (\mu > 0;\ z \in C), \tag{1.4}$$

so that, by comparing the definitions (1.1) and (1.4),

$$J_\nu(z) = \left(\frac{z}{2}\right)^\nu\ J_\nu^1\left(\frac{z^2}{4}\right). \tag{1.5}$$

Further, another generalization of the Bessel function defind by Pathak [6] as follows:

$$J_{\nu,\lambda}^\mu(z) = \sum_{m=0}^{\infty} \frac{(-1)^m\ (z/2)^{\nu+2\lambda+2m}}{\Gamma(\lambda+m+1)\ \Gamma(\nu+\lambda+\mu m+1)}, \tag{1.6}$$

where $z \in C\backslash(-\infty,\ 0];\ \mu > 0,\ \nu,\ \lambda \in C.$

So that

$$J_{\nu,0}^1(z) = J_\nu(z) \tag{1.7}$$

and

$$J_{\nu,0}^\mu(z) = \left(\frac{z}{2}\right)^\nu J_\nu^\mu\left(\frac{z^2}{4}\right) \qquad (\ \mu \in \Re^+\ ). \tag{1.8}$$

The generalization of the generalized hypergeometric series ${}_pF_q$ (1.12) is due to Fox [4] and Wright ([12], [14], [15]) who studied the asymptotic expansion of the generalized (Wright) hypergeometric function defined by (see [11, p.21]; see also [9])

$${}_p\Psi_q\left[\begin{matrix}(\alpha_1\ ,\ A_1), & ....., & (\alpha_p\ ,\ A_p); \\ & & & z \\ (\beta_1\ ,\ B_1), & ....., & (\beta_q\ ,\ B_q);\end{matrix}\right] = \sum_{k=0}^{\infty} \frac{\prod_{j=1}^{p}\ \Gamma(\alpha_j+A_jk)}{\prod_{j=1}^{q}\ \Gamma(\beta_j+B_jk)}\ \frac{z^k}{k!}\ , \tag{1.9}$$

where the coefficients $A_1, \cdots, A_p$ and $B_1, \cdots, B_q$ are positive real numbers such that

(i) $1+\sum_{j=1}^{q} B_j - \sum_{j=1}^{p} A_j > 0$ and $0 < |z| < \infty;\ z \neq 0.$

(1.10a)

(ii) $1+\sum_{j=1}^{q} B_j-\sum_{j=1}^{p} A_j = 0$ and $0 < |z| < {A_1}^{-A_1} \dots {A_p}^{-A_p} {B_1}^{B_1} \dots {B_q}^{B_q}$.

(1.10b)

A special case of (1.9) is

$$ {}_p\Psi_q \left[\begin{matrix} (\alpha_1\ ,\ 1), & ....., & (\alpha_p\ ,\ 1); \\ (\beta_1\ ,\ 1), & ....., & (\beta_q\ ,\ 1); \end{matrix} \; z \right] = \frac{\prod_{j=1}^{p} \Gamma(\alpha_j)}{\prod_{j=1}^{q} \Gamma(\beta_j)} \, {}_pF_q \left[\begin{matrix} \alpha_1, & ....., & \alpha_p\ ; \\ \beta_1, & ....., & \beta_q\ ; \end{matrix} \; z \right], \tag{1.11} $$

where ${}_pF_q$ is the generalized hypergeometric series defined by (see [10, section 1.5])

$$ {}_pF_q \left[\begin{matrix} \alpha_1, & ....., & \alpha_p\ ; \\ \beta_1, & ....., & \beta_q\ ; \end{matrix} \; z \right] = \sum_{n=0}^{\infty} \frac{(\alpha_1)_n \cdots (\alpha_p)_n}{(\beta_1)_n \cdots (\beta_q)_n} \frac{z^n}{n!} $$

$$ = {}_pF_q(\alpha_1, \cdots, \alpha_p;\ \beta_1, \cdots \beta_q;\ z), \tag{1.12} $$

where $(\lambda)_n$ is called the Pochhammer's symbol [7].

2. Result Required

For our present investigation, the following interesting result due to MacRobert [5] will be required.

$$ \int_0^1 x^{\alpha-1}(1-x)^{\beta-1}[ax+b(1-x)]^{-\alpha-\beta}\, dx = \frac{1}{a^\alpha b^\beta} \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)}, \tag{2.1} $$

provided $\Re(\alpha) > 0$ and $\Re(\beta) > 0$ and a and b are non-zero constants and the expression $ax + b(1-x)$, where $0 \le x \le 1$, is non-zero.

3. Main Result

The following integral formula holds true: For $\alpha, \beta, \nu, \lambda, \mu \in C$ with $\Re(\lambda) > -1$, $\Re(\nu+\lambda) > -1$, $\Re(\alpha+\nu+2\lambda) > 0$, $\Re(\beta+\nu+2\lambda) > 0$ and the expression $ax + b(1-x)$, where $0 \le x \le 1$, is non-zero.

$$ \int_0^1 x^{\alpha-1}(1-x)^{\beta-1}[ax+b(1-x)]^{-\alpha-\beta} J^{\mu}_{\nu,\lambda} \left[\frac{2abx(1-x)}{[ax+b(1-x)]^2} \right] dx $$

$$ = \frac{1}{a^\alpha b^\beta} {}_3\Psi_3 \left[\begin{matrix} (\alpha+\nu+2\lambda, 2), & (\beta+\nu+2\lambda, 2), & (1,1) & ; \\ (\lambda+1, 1), & (\nu+\lambda+1, \mu), & (\alpha+\beta+2\nu+4\lambda, 4) & ; \end{matrix} \; -1 \right], \tag{3.1} $$

where $J^{\mu}_{\nu,\lambda}(z)$ is the generalized Bessel-Maitland function (1.6).
Proof:
In order to derive the main integral (3.1), we denote the left hand side of (3.1) by I, expressing $J^{\mu}_{\nu,\lambda}$ as a series with the help of definition (1.6), we have

$$I=\int_0^1 x^{\alpha-1}(1-x)^{\beta-1}[ax+b(1-x)]^{-\alpha-\beta}$$

$$\times\sum_{m=0}^{\infty}\frac{(-1)^m}{\Gamma(\lambda+m+1)\Gamma(\nu+\lambda+\mu m+1)}\left[\frac{abx(1-x)}{[ax+b(1-x)]^2}\right]^{\nu+2\lambda+2m}dx, \tag{3.2}$$

changing the order of integration and summation, which is verified by uniform convergence of the involved series under the given conditions, we get

$$I=(ab)^{\nu+2\lambda}\sum_{m=0}^{\infty}\frac{(-1)^m(ab)^{2m}}{\Gamma(\lambda+m+1)\Gamma(\nu+\lambda+\mu m+1)}$$

$$\times\int_0^1 x^{\alpha+\nu+2\lambda+2m-1}(1-x)^{\beta+\nu+2\lambda+2m-1}[ax+b(1-x)]^{-\alpha-\beta-2\nu-4\lambda-4m}\,dx. \tag{3.3}$$

Evaluating the above integral with the help of (2.1) and after little simplification, we have

$$I=\frac{1}{a^{\alpha}b^{\beta}}\frac{\Gamma(\alpha+\nu+2\lambda+2m)\Gamma(\beta+\nu+2\lambda+2m)\Gamma(m+1)(-1)^m}{\Gamma(\lambda+m+1)\Gamma(\nu+\lambda+\mu m+1)\Gamma(\alpha+\beta+2\nu+4\lambda+4m)m!}, \tag{3.4}$$

which upon using (1.9) yields (3.1). This completes the proof of our main result.
4. Variation of (3.1): Let the conditions of our main result be satisfied. Then the following integral formula holds true:

$$\int_0^1 x^{\alpha-1}(1-x)^{\beta-1}[ax+b(1-x)]^{-\alpha-\beta}J^{\mu}_{\nu,\lambda}\left[\frac{2abx(1-x)}{[ax+b(1-x)]^2}\right]dx$$

$$=\frac{1}{a^{\alpha}b^{\beta}}\frac{\Gamma(\alpha+\nu+2\lambda)\Gamma(\beta+\nu+2\lambda)}{\Gamma(\lambda+1)\Gamma(\nu+\lambda+1)\Gamma(\alpha+\beta+2\nu+4\lambda)}$$

$$\times{}_5F_{\mu+5}\left[\begin{matrix}\Delta(2;\alpha+\nu+2\lambda), & \Delta(2;\beta+\nu+2\lambda), & 1 & ; \\ & & & & -\frac{1}{16\mu^{\mu}} \\ \Delta(\mu;\nu+\lambda+1), & \Delta(4;\alpha+\beta+2\nu+4\lambda), & \lambda+1 & ; \end{matrix}\right], \tag{4.1}$$

where $\Delta(k;\lambda)$ abbreviates the arrangement of k parameters $\frac{\lambda}{k}, \frac{\lambda+1}{k}, ..., \frac{\lambda+k-1}{k}$, $k \geq 1$.
Proof:

By writing the right hand side of (3.1) in the original summation and using the result

$$(\lambda)_{kn} = \left(\frac{\lambda}{k}\right)_n, \left(\frac{\lambda+1}{k}\right)_n, ..., \left(\frac{\lambda+k-1}{k}\right)_n,$$

after a little simplification, we get the required result (4.1).

5. Special Cases

In this section, we derive certain new integral formulas as special cases of our main result.

(i). On taking $\lambda = 0$ in (3.1) and then using (1.8), after a little simplification, we get

$$\int_0^1 x^{\alpha+\nu-1}(1-x)^{\beta+\nu-1}[ax+b(1-x)]^{-\alpha-\beta-2\nu} J_\nu^\mu \left[\frac{a^2b^2x^2(1-x)^2}{[ax+b(1-x)]^4}\right] dx$$

$$= \frac{1}{a^{\alpha+\nu}b^{\beta+\nu}} {}_2\Psi_2 \left[\begin{array}{lll} (\alpha+\nu,2), & (\beta+\nu,2) & ; \\ & & -1 \\ (\nu+1,\mu), & (\alpha+\beta+2\nu,4) & ; \end{array}\right], \tag{5.1}$$

where $\Re(\nu) > -1$, $\Re(\alpha+\nu) > 0$ and $\Re(\beta+\nu) > 0$.

(ii). On taking $\mu = 1$ in (5.1) and then using (1.5), we get

$$\int_0^1 x^{\alpha-1}(1-x)^{\beta-1}[ax+b(1-x)]^{-\alpha-\beta} J_\nu \left[\frac{2abx(1-x)}{[ax+b(1-x)]^2}\right] dx$$

$$= \frac{1}{a^{\alpha}b^{\beta}} {}_2\Psi_2 \left[\begin{array}{lll} (\alpha+\nu,2), & (\beta+\nu,2) & ; \\ & & -1 \\ (\nu+1,1), & (\alpha+\beta+2\nu,4) & ; \end{array}\right], \tag{5.2}$$

where $\Re(\nu) > -1$, $\Re(\alpha+\nu) > 0$ and $\Re(\beta+\nu) > 0$.

(iii). On taking $\nu = \frac{1}{2}$ in (5.2) and then using (1.3), after a little simplification, we get

$$\int_0^1 x^{\alpha-\frac{3}{2}}(1-x)^{\beta-\frac{3}{2}}[ax+b(1-x)]^{1-\alpha-\beta} \sin\left[\frac{2abx(1-x)}{[ax+b(1-x)]^2}\right] dx$$

$$= \frac{\sqrt{\pi ab}}{a^\alpha b^\beta} {}_2\Psi_2 \left[\begin{array}{ccc} (\alpha+\frac{1}{2},2), & (\beta+\frac{1}{2},2) & ; \\ & & & -1 \\ (\frac{3}{2},1), & (\alpha+\beta+1,4) & ; \end{array} \right], \quad (5.3)$$

where $\Re(\alpha+\frac{1}{2})>0$ and $\Re(\beta+\frac{1}{2})>0$.

(iv). On taking $\nu=-\frac{1}{2}$ in (5.2) and then using (1.2), after a little simplification, we get

$$\int_0^1 x^{\alpha-\frac{3}{2}}(1-x)^{\beta-\frac{3}{2}}[ax+b(1-x)]^{1-\alpha-\beta}\cos\left[\frac{2abx(1-x)}{[ax+b(1-x)]^2}\right]\,dx$$

$$= \frac{\sqrt{\pi ab}}{a^\alpha b^\beta} {}_2\Psi_2 \left[\begin{array}{ccc} (\alpha-\frac{1}{2},2), & (\beta-\frac{1}{2},2) & ; \\ & & & -1 \\ (\frac{1}{2},1), & (\alpha+\beta-1,4) & ; \end{array} \right], \quad (5.4)$$

where $\Re(\alpha-\frac{1}{2})>0$ and $\Re(\beta-\frac{1}{2})>0$.

(v). On taking $\lambda=0$ in (4.1) and then using (1.8), after a little simplification, we get

$$\int_0^1 x^{\alpha+\nu-1}(1-x)^{\beta+\nu-1}[ax+b(1-x)]^{-\alpha-\beta-2\nu}J_\nu^\mu\left[\frac{a^2b^2x^2(1-x)^2}{[ax+b(1-x)]^4}\right]\,dx$$

$$= \frac{1}{a^{\alpha+\nu}b^{\beta+\nu}}\frac{\Gamma(\alpha+\nu)\Gamma(\beta+\nu)}{\Gamma(\nu+1)\Gamma(\alpha+\beta+2\nu)}$$

$$\times {}_4F_{\mu+4}\left[\begin{array}{ccc} \Delta(2;\alpha+\nu), & \Delta(2;\beta+\nu) & ; \\ & & & -\frac{1}{16\mu^\mu} \\ \Delta(\mu;\nu+1), & \Delta(4;\alpha+\beta+2\nu), & ; \end{array} \right], \quad (5.5)$$

where μ is positive integer, $\Re(\nu)>-1$, $\Re(\alpha+\nu)>0$ and $\Re(\beta+\nu)>0$.

(vi). On taking $\mu=1$ in (5.5) and then using (1.5), after a little simplification, we get

$$\int_0^1 x^{\alpha-1}(1-x)^{\beta-1}[ax+b(1-x)]^{-\alpha-\beta}J_\nu\left[\frac{2abx(1-x)}{[ax+b(1-x)]^42}\right]\,dx$$

$$= \frac{1}{a^\alpha b^\beta}\frac{\Gamma(\alpha+\nu)\Gamma(\beta+\nu)}{\Gamma(\nu+1)\Gamma(\alpha+\beta+2\nu)}$$

$$\times {}_4F_5\left[\begin{array}{ccc} \Delta(2;\alpha+\nu), & \Delta(2;\beta+\nu) & ; \\ & & & -\frac{1}{16} \\ \nu+1, & \Delta(4;\alpha+\beta+2\nu) & ; \end{array} \right], \quad (5.6)$$

where $\Re(\nu) > -1$, $\Re(\alpha+\nu) > 0$ and $\Re(\beta+\nu) > 0$.

(vii). On taking $\nu = \frac{1}{2}$ in (5.6) and then using (1.3), after a little simplification, we get

$$\int_0^1 x^{\alpha-\frac{3}{2}}(1-x)^{\beta-\frac{3}{2}}[ax+b(1-x)]^{1-\alpha-\beta} \sin\left[\frac{2abx(1-x)}{[ax+b(1-x)]^2}\right] dx$$

$$= \frac{2\sqrt{ab}}{a^\alpha b^\beta} \frac{\Gamma(\alpha+1/2)\Gamma(\beta+1/2)}{\Gamma(\alpha+\beta+1)} {}_4F_5\left[\begin{array}{ccc} \Delta(2;\alpha+\frac{1}{2}), & \Delta(2;\beta+\frac{1}{2}) & ; \\ & & -\frac{1}{16} \\ \frac{3}{2}, & \Delta(4;\alpha+\beta+1) & ; \end{array}\right], \tag{5.7}$$

where $\Re(\alpha+\frac{1}{2}) > 0$ and $\Re(\beta+\frac{1}{2}) > 0$.

(viii). Further on taking $\nu = -\frac{1}{2}$ in (5.6) and then using (1.2), after a little simplification, we get

$$\int_0^1 x^{\alpha-\frac{3}{2}}(1-x)^{\beta-\frac{3}{2}}[ax+b(1-x)]^{1-\alpha-\beta} \cos\left[\frac{2abx(1-x)}{[ax+b(1-x)]^2}\right] dx$$

$$= \frac{\sqrt{ab}}{a^\alpha b^\beta} \frac{\Gamma(\alpha-1/2)\Gamma(\beta-1/2)}{\Gamma(\alpha+\beta-1)} {}_4F_5\left[\begin{array}{ccc} \Delta(2;\alpha-\frac{1}{2}), & \Delta(2;\beta-\frac{1}{2}) & ; \\ & & -\frac{1}{16} \\ \frac{1}{2}, & \Delta(4;\alpha+\beta-1) & ; \end{array}\right], \tag{5.8}$$

where $\Re(\alpha-\frac{1}{2}) > 0$ and $\Re(\beta-\frac{1}{2}) > 0$.

6. Concluding Remarks:

In the present investigation, we have established a unified integral formula involving generalized Bessel-Maitland function, which is expressed in terms of the generalized (Wright) hypergeometric function. Some other integral formulas as special cases of our main results are also considered. It can be easily seen that the Mittag-Leffler function is a special case of Bessel-Maitland function. Therefore, the results presented in this paper are easily converted in terms of Mittag-Leffler function after a suitable parametric replacement.

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