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OSCILLATION OF NONLINEAR NEUTRAL PARTIAL DIFFERENTIAL EQUATIONS WITH DAMPING TERM

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Abstract: In this paper, we consider the oscillation of forced solutions of nonlinear impulsive neutral partial differential equations with damping term. Sufficient conditions are obtained for the oscillation of solutions using impulsive differential inequalities with two boundary conditions. Example is given to illustrate our main result.

Keywords and Phrases: Neutral partial differential equations, Oscillation, Impulse, Damping term.

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1. Introduction

The theory of partial functional differential equations can be applied to many fields, such as biology, population growth, engineering, control theory, physics and chemistry. Oscillation theory of differential equations originated by C. Sturm [20] in 1836, and for partial differential equations by P. Hartman and A. Wintner [7] in 1955. Pioneer work on oscillation of impulsive delay differential equations [6] was published in 1989 and its results were included in monograph [8]. In 1991, the first work done in [2] on impulsive partial differential equations.

Many authors studied the oscillation of partial differential equations with or without impulsive neutral type, see [1,3-5,9-14,16-19,21,23-25,27] and monographs [22,26]. To the best of our knowledge, there is little work reported on the oscillation of second order impulsive partial functional differential equation with damping. Motivated by this observation, in this paper we focus our attention on oscillation of forced nonlinear impulsive neutral partial differential equations with damping term

$$\left.\begin{array}{l} \dfrac{\partial}{\partial t}\left[r(t)\dfrac{\partial}{\partial t}\left(u(x,t)+c(t)u(x,\tau(t))\right)\right]+p(t)\dfrac{\partial}{\partial t}\left(u(x,t)+c(t)u(x,\tau(t))\right) \\ +q(x,t)f(u(x,t))+\sum\limits_{i=1}^{n}q_i(x,t)f_i(u(x,\sigma_i(t)))=a(t)h(u(x,t))\Delta u(x,t) \\ +\sum\limits_{j=1}^{m}b_j(t)h_j(u(x,\rho_j(t)))\Delta u(x,\rho_j(t))+g(x,t), \\ t\neq t_k,\ (x,t)\in\Omega\times[0,+\infty)\equiv G \\ u(x,t_k^+)=\ \alpha_k\left(x,t_k,u(x,t_k)\right), \\ u_t(x,t_k^+)=\ \beta_k\left(x,t_k,u_t(x,t_k)\right), \quad t=t_k, \quad k=1,2,.... \end{array}\right\} \quad (E)$$

with the boundary conditions

$$u=\ 0, \quad (x,t)\in\partial\Omega\times[0,+\infty) \quad (B1)$$

$$\frac{\partial u}{\partial\gamma}+\mu(x,t)u=\ 0, \quad (x,t)\in\partial\Omega\times[0,+\infty) \quad (B2)$$

and the initial condition

$$u(x,t)=\Phi(x,t), \qquad \frac{\partial u(x,t)}{\partial t}=\Psi(x,t), \qquad (x,t)\in\Omega\times[-\delta,0].$$

Here $\Omega\subset\mathbb{R}^N$ is a bounded domain with boundary $\partial\Omega$ smooth, Δ is the Laplacian in the Euclidean N-space $\mathbb{R}^N$ and γ is a unit exterior normal vector of $\partial\Omega$, $\delta=\max\{\tau(t),\sigma_i(t),\rho_j(t)\}$, $\Phi(x,t)\in C^2([-\delta,0]\times\Omega,\mathbb{R})$, $\Psi(x,t)\in C^1([-\delta,0]\times\Omega,\mathbb{R})$, $\mu(x,t)\in C(\partial\Omega\times[0,+\infty),[0,+\infty))$.

We assume that the following hypotheses (H) hold:

(H1) $r(t)\in C^1([0,+\infty),(0,+\infty))$, $r'(t)\geq 0$, $p(t)\in C([0,+\infty),\mathbb{R})$, $\int_{t_0}^{+\infty}\dfrac{1}{R(s)}ds=+\infty$, where $R(t)=\exp\left(\int_{t_0}^{t}\dfrac{r'(s)+p(s)}{r(s)}ds\right)$, $c(t)\in C^2([0,+\infty),[0,+\infty))$, $a(t),b_j(t)\in PC([0,+\infty),[0,+\infty))$, $\tau(t)$, $\sigma_i(t),\rho_j(t)$ are positive constants, $q(x,t)$, $q_j(x,t)\in C\left(\bar{\Omega}\times[0,+\infty),[0,+\infty)\right)$, $q(t)=\min\limits_{x\in\bar{\Omega}} q(x,t)$, $q_i(t)=\min\limits_{x\in\bar{\Omega}} q_i(x,t), i=1,2,\cdots,n$, where PC denote the class of functions which are piecewise continuous in t with discontinuities of first kind only at $t=t_k, k=1,2,\cdots$ and left continuous at $t=t_k$, $k=1,2,\cdots$.

(H2) $h(u),\ h_j(u) \in C^1(\mathbb{R},\mathbb{R}),\ f(u),\ f_j(u) \in C(\mathbb{R},\mathbb{R})$ is convex in $[0,+\infty)$, $uf(u) > 0$ and $\frac{f(u)}{u} \geq \epsilon > 0$, $\frac{f_j(u)}{u} \geq \epsilon_j > 0$ are positive constant for $u \neq 0, uh'(u) \geq 0,\ uh'_j(u) \geq 0, h(u)\mu(x,t) \geq 0, h_j(u)\mu(x,\rho_j(t)) \geq 0,\ j = 1,2,\cdots,m,\ 0 < t_1 < \cdots < t_k < \cdots,\ \lim\limits_{t\to+\infty} t_k = +\infty,\ g(x,t) \in PC\left(\bar{\Omega}\times[0,+\infty),\mathbb{R}\right)$, $\int_\Omega g(x,t)dx \leq 0$.

(H3) $u(x,t)$ and their derivatives $u_t(x,t)$ are piecewise continuous in t with discontinuities of first kind only at $t = t_k,\quad k = 1,2,\cdots$, and left continuous at $t = t_k$, $u(x,t_k) = u(x,t_k^-)$, $u_t(x,t_k) = u_t(x,t_k^-)$, $k = 1,2,\cdots$.

(H4) $\alpha_k\left(x,t_k,u(x,t_k)\right),\ \beta_k\left(x,t_k,u_t(x,t_k)\right) \in PC\left([0,+\infty)\times\bar{\Omega}\times\mathbb{R},\ \mathbb{R}\right),\quad k = 1,2,\cdots$, and there exist positive constants $a_k,\ a_k^*,\ b_k,\ b_k^*$ with $b_k \leq a_k^*$ such that for $k = 1,2,\cdots$,

$$a_k^* \leq \frac{\alpha_k\left(x,t_k,u(x,t_k)\right)}{u(x,t_k)} \leq a_k,$$
$$b_k^* \leq \frac{\beta_k\left(x,t_k,u_t(x,t_k)\right)}{u_t(x,t_k)} \leq b_k.$$

Let us construct the sequence $\{\bar{t}_k\} = \{t_k\}\cup\{t_{k\tau}\}\cup\{t_{k\sigma_i}\}\cup\left\{t_{k\rho_j}\right\}$, where $t_{k\tau} = t_k+\tau$, $t_{k\sigma_i} = t_k+\sigma_i$, $t_{k\rho_j} = t_k+\rho_j$ and $\bar{t}_k < \bar{t}_{k+1}$, $i = 1,2,\cdots,n,\ j = 1,2,\cdots,m,\ k = 1,2,\cdots$.

Definition 1.1. *By a solution of problem* $(E),(B1)$ $((E),(B2))$ *with initial condition, we mean that any function* $u(x,t)$ *for which the following conditions are valid:*

(1) If $-\delta \leq t \leq 0$, *then* $u(x,t) = \Phi(x,t)$, $\dfrac{\partial u(x,t)}{\partial t} = \Psi(x,t)$.

(2) If $0 \leq t \leq \bar{t}_1 = t_1$, *then* $u(x,t)$ *coincides with the solution of the problem* (E) *and* $(B1)$ $((B2))$ *with initial condition.*

(3) If $\bar{t}_k < t \leq \bar{t}_{k+1},\ \bar{t}_k \in \{t_k\}\setminus\left(\{t_{k\tau}\}\cup\{t_{k\sigma_i}\}\cup\left\{t_{k\rho_j}\right\}\right)$, *then* $u(x,t)$ *coincides with the solution of the problem* (E) *and* $(B1)$ $((B2))$.

(4) If $\bar{t}_k < t \leq \bar{t}_{k+1},\ \bar{t}_k \in \{t_{k\tau}\}\cup\{t_{\sigma_i}\}\cup\left\{t_{k\rho_j}\right\}$, *then* $u(x,t)$ *satisfies* $(B1)$ $((B2))$ *and coincides with the solution of the problem.*

$$\frac{\partial}{\partial t}\left[r(t)\frac{\partial}{\partial t}\left(u(x,t^+)+c(t)u(x,(\tau(t))^+)\right)\right]+p(t)\frac{\partial}{\partial t}\left(u(x,t^+)+c(t)u(x,(\tau(t))^+)\right)$$
$$+q(x,t)f(u(x,t^+))+\sum_{i=1}^{n}q_i(x,t)f_i(u(x,(\sigma_i(t))^+))=a(t)h(u(x,t^+))\Delta u(x,t^+)$$
$$+\sum_{j=1}^{m}b_j(t)h_j(u(x,(\rho_j(t))^+))\Delta u(x,(\rho_j(t))^+)+g(x,t^+),\quad t\neq t_k,$$
$$(x,t)\in\Omega\times[0,+\infty)\equiv G$$

$u(x,\bar{t}_k^+) = u(x,\bar{t}_k),\ \ u_t(x,\bar{t}_k^+) = u_t(x,\bar{t}_k),$ *for* $\bar{t}_k \in \left(\{t_{k\tau}\}\cup\{t_{k\sigma_i}\}\cup\{t_{k\rho_j}\}\right)\cap \{t_k\}$,
or
$u(x,\bar{t}_k^+) = \alpha_{k_s}(x,\bar{t}_k,u(x,\bar{t}_k)),\ \ u_t(x,\bar{t}_k^+) = \beta_{k_s}(x,\bar{t}_k,u_t(x,\bar{t}_k)),$
for $\bar{t}_k\in\left(\{t_{k\tau}\}\cup\{t_{k\sigma_i}\}\cup\{t_{k\rho_j}\}\right)\cap\{t_k\}$.

Here the number k_s *is determined by the equality* $\bar{t}_k = t_{k_s}$.

We introduce the notations:

$$\Gamma_k = \{(x,t) : t\in(t_k,t_{k+1}), x\in\Omega\}; \qquad \Gamma=\cup_{k=0}^{\infty}\Gamma_k,$$
$$\bar{\Gamma}_k = \left\{(x,t) : t\in(t_k,t_{k+1}), x\in\bar{\Omega}\right\}; \qquad \bar{\Gamma}=\cup_{k=0}^{\infty}\bar{\Gamma}_k.$$

For each positive solution $u(x,t)$ of $(E),\ (B1)$ $((B2))$, we associate the function $v(t)$ defined by

$$v(t)=\int_{\Omega}u(x,t)dx, \qquad g_0=1-c(t), \qquad g_1=1-c(\sigma_i(t)).$$

Definition 1.2. *The solution* $u\in C^2(\Gamma)\cap C^1(\bar{\Gamma})$ *of problem* $(E),(B1)$ $((B2))$ *is called non-oscillatory in the domain* G *if it is either eventually positive or eventually negative. Otherwise, it is called oscillatory.*

This paper is organized as follows: Section 2, deals with the oscillatory properties of solutions for the problem (E) and $(B1)$. In Section 3, we discuss the oscillatory properties of solutions for the problem (E) and $(B2)$. Section 4 presents some example to illustrate the main result.

2. Oscillation properties of the problem (E) and $(B1)$

To prove the main result, we need the following lemmas.

Lemma 2.1. *Let* $u\in C^2(\Gamma)\cap C^1(\bar{\Gamma})$ *be a positive solution of the problem* $(E),\ (B1)$

in G, then function $z(t)$ satisfies the impulsive differential inequality

$$\left.\begin{aligned} &[r(t)z'(t)]' + p(t)z'(t) + \epsilon g_0 q(t)z(t) + \sum_{i=1}^{n} \epsilon_i g_1 q_i(t) z(\sigma_i(t)) \le 0,\ t \ne t_k \\ &a_k^* \le \frac{z(t_k^+)}{z(t_k)} \le a_k \\ &b_k^* \le \frac{z'(t_k^+)}{z'(t_k)} \le b_k,\ t = t_k, \quad k = 1, 2, ... \end{aligned}\right\} \tag{2.1}$$

where $z(t) = v(t) + c(t)v(\tau(t))$.

Proof Let $u(x,t)$ be a positive solution of the problem (E), $(B1)$ in G. Without loss of generality, we may assume that there exists a $T > 0,\ t_0 > T$ such that $u(x,t) > 0,\ u(x,\tau(t)) > 0,\ u(x,\sigma_i(t)) > 0,\ i = 1, 2, \cdots, n,\ u(x,\rho_j(t)) > 0,\ j = 1, 2, \cdots, m$ for any $(x,t) \in \Omega \times [t_0, +\infty)$.

For $t \ge t_0,\ t \ne t_k,\ k = 1, 2, \cdots$, integrating (E) with respect to x over the domain Ω yields

$$\left.\begin{aligned} &\frac{d}{dt}\left[r(t)\frac{d}{dt}\left(\int_\Omega u(x,t)dx + \int_\Omega c(t)u(x,\tau(t))dx\right)\right] \\ &+p(t)\frac{d}{dt}\left(\int_\Omega u(x,t)dx + \int_\Omega c(t)u(x,\tau(t))dx\right) \\ &+\int_\Omega q(x,t)f(u(x,t))dx + \sum_{i=1}^{n}\int_\Omega q_i(x,t)f_i(u(x,\sigma_i(t)))dx \\ &= \int_\Omega a(t)h(u(x,t))\Delta u(x,t)dx + \sum_{j=1}^{m}\int_\Omega b_j(t)h_j(u(x,\rho_j(t)))\Delta u(x,\rho_j(t))dx \\ &\quad + \int_\Omega g(x,t)dx \end{aligned}\right\} \tag{2.2}$$

By Green's formula, and the boundary condition $(B1)$, we have

$$\begin{aligned} \int_\Omega h(u(x,t))\Delta u(x,t)dx &= \int_{\partial\Omega} h(u(x,t))\frac{\partial u(x,t)}{\partial \gamma}dS - \int_\Omega h'(u(x,t))\left|grad\ u\right|^2 dx \\ &= -\int_\Omega h'(u(x,t))\left|grad\ u\right|^2 dx \le 0, \end{aligned} \tag{2.3}$$

and for $j = 1, 2, \cdots, m$

$$\begin{aligned} \int_\Omega h_j(u(x,\rho_j(t)))\Delta u(x,\rho_j(t))dx &= \int_{\partial\Omega} h_j(u(x,\rho_j(t)))\frac{\partial u(x,\rho_j(t))}{\partial \gamma}dS \\ &\quad - \int_\Omega h_j'(u(x,\rho_j(t)))\left|grad\ u\right|^2 dx \\ &= -\int_\Omega h_j'(u(x,\rho_j(t)))\left|grad\ u\right|^2 dx \le 0, \end{aligned} \tag{2.4}$$

where dS is the surface element on $\partial\Omega$. Moreover using Jensen's inequality, from $(H2)$ and assumptions it follows that

$$\begin{aligned}\int_\Omega q(x,t)f(u(x,t))dx &\geq q(t)\int_\Omega f(u(x,t))dx \\ &\geq \epsilon q(t)\int_\Omega u(x,t)dx \\ &\geq \epsilon q(t)v(t) \qquad (2.5)\end{aligned}$$

and for $i = 1, 2, \cdots, n$

$$\int_\Omega q_i(x,t)f_i(u(x,\sigma_i(t)))dx \geq \epsilon_i q_i(t)v(\sigma_i(t)). \qquad (2.6)$$

In view of (2.2)-(2.6), we obtain

$$\frac{d}{dt}\left[r(t)\frac{d}{dt}\left(v(t)+c(t)v(\tau(t))\right)\right] + p(t)\frac{d}{dt}\left(v(t)+c(t)v(\tau(t))\right) + \epsilon q(t)v(t) + \sum_{i=1}^{n}\epsilon_i q_i(t)v(\sigma_i(t)) \leq 0, \quad t \neq t_k.$$

Set $z(t) = v(t) + c(t)v(\tau(t))$. Then

$$(r(t)z'(t))' + p(t)z'(t) + \epsilon q(t)v(t) + \sum_{i=1}^{n}\epsilon_i q_i(t)v(\sigma_i(t)) \leq 0, \quad t \neq t_k. \qquad (2.7)$$

It is easy to obtain that $z(t) > 0$ for $t \geq t_0$. Next we prove that $z'(t) > 0$ for $t \geq t_1$. In fact assume the contrary, there exists $T \geq t_1$ such that $z'(T) \leq 0$.

$$\begin{aligned}(r(t)z'(t))' + p(t)z'(t) &\leq 0, \quad t \geq t_1 \\ r(t)z''(t) + (r'(t)+p(t))\,z'(t) &\leq 0, \quad t \geq t_1. \qquad (2.8)\end{aligned}$$

From $(H1)$, we have $R'(t) = R(t)\left(\frac{r'(t)+p(t)}{r(t)}\right)$ and $R(t) > 0$, $R'(t) \geq 0$ for $t \geq t_1$. Thus we multiply $\frac{R(t)}{r(t)}$ on both sides of (2.8), we have

$$R(t)z''(t) + R'(t)z'(t) = (R(t)z'(t))' \leq 0, \quad t \geq t_1. \qquad (2.9)$$

From (2.9), we have $R(t)z'(t) \leq R(T)z'(T) \leq 0,\ t \geq T$. Thus

$$\int_T^t z'(s)ds \leq \int_T^t \frac{R(T)z'(T)}{R(s)}ds, \quad t \geq T$$
$$z(t) \leq z(T) + R(T)z'(T)\int_T^t \frac{ds}{R(s)}, \quad t \geq T.$$

From the hypotheses $(H1)$, we have $\lim_{t\to+\infty} z(t) = -\infty$. This contradicts that $z(t) > 0$ for $t \geq 0$. Thus $z'(t) > 0$ and $\tau(t) \leq t$ for $t \geq t_1$, we have

$$\begin{aligned} v(t) =& \ z(t) - c(t)v(\tau(t)) \\ v(\tau(t)) =& \ z(\tau(t)) - c(\tau(t))v(\tau(\tau(t))) \\ v(t) =& \ z(t) - c(t)z(\tau(t)) - c(t)c(\tau(t))v(\tau(\tau(t))) \\ \geq& \ z(t)(1 - c(t)) \\ v(t) \geq& \ g_0 z(t) \end{aligned}$$

and

$$\begin{aligned} v(\sigma_i(t)) \geq& \ z(\sigma_i(t))(1 - c(\sigma_i(t))) \\ v(\sigma_i(t)) \geq& \ g_1 z(\sigma_i(t)). \end{aligned}$$

Therefore from (2.7), we have

$$(r(t)z'(t))' + p(t)z'(t) + g_0\epsilon q(t)z(t) + \sum_{i=1}^{n} g_1\epsilon_i q_i(t)z(\sigma_i(t)) \leq 0, \quad t \geq t_1, \quad t \neq t_k.$$

For $t \geq t_0,\ t = t_k,\ k = 1, 2, \cdots$, integrating (E) with respect to x over the domain Ω from $(H4)$, we obtain

$$a_k^* \leq \frac{u(x, t_k^+)}{u(x, t_k^+)} \leq a_k, \qquad b_k^* \leq \frac{u_t(x, t_k^+)}{u_t(x, t_k)} \leq b_k.$$

According to $v(t) = \int_\Omega u(x,t)dx$, we have

$$a_k^* \leq \frac{v(t_k^+)}{v(t_k)} \leq a_k, \qquad b_k^* \leq \frac{v'(t_k^+)}{v'(t_k)} \leq b_k.$$

Because $z(t) = v(t) + c(t)v(\tau(t))$, we obtain

$$a_k^* \leq \frac{z(t_k^+)}{z(t_k)} \leq a_k, \qquad b_k^* \leq \frac{z'(t_k^+)}{z'(t_k)} \leq b_k.$$

Therefore $z(t)$ is an eventually positive solution of (2.1). This contradicts the hypothesis and completes the proof.

Lemma 2.1. *Assume that*

(A1) the sequence $\{t_k\}$ *satisfies* $0 < t_0 < t_1 < ...,\ \lim_{k\to+\infty} t_k = +\infty$*;*

(A2) $m(t) \in PC^1[[0,+\infty),\mathbb{R}]$ *is left continuous at* t_k *for* $k = 1, 2, \cdots$ *;*

(A3) for $k = 1, 2, \cdots$ *, and* $t \geq t_0$,

$$\begin{aligned} m'(t) &\leq l(t)m(t) + q(t), \quad t \neq t_k, \\ m(t_k^+) &\leq d_k m(t_k) + e_k, \end{aligned}$$

where $l(t),\ q(t) \in C([0,+\infty),\mathbb{R})$, $d_k \geq 0$ *and* e_k *are constants. PC denote the class of piecewise continuous function from* $[0,+\infty)$ *to* $\mathbb{R}$, *with discontinuities of the first kind only at* $t = t_k, \quad k = 1, 2, \cdots$.

Then

$$\begin{aligned} m(t) \ &\leq m(t_0) \prod_{t_0<t_k<t} d_k exp\left(\int_{t_0}^t l(s)ds\right) + \int_{t_0}^t \prod_{s<t_k<t} d_k exp\left(\int_s^t l(r)dr\right) q(s)ds \\ &+ \sum_{t_0<t_k<t} \prod_{t_k<t_j<t} d_j exp\left(\int_{t_k}^t l(s)ds\right) e_k. \end{aligned}$$

Proof The proof of the lemma can be found in [8].

Lemma 2.3. *Let* $z(t)$ *be an eventually positive (negative) solution of the differential inequality (2.1). Assume that there exists* $T \geq t_0$ *such that* $z(t) > 0$ $(z(t) < 0)$ *for* $t \geq T$. *If*

$$\lim_{t\to+\infty} \int_{t_0}^t \prod_{t_0<t_k<s} \frac{b_k^*}{a_k} ds = +\infty \tag{2.10}$$

hold, then $z'(t) \geq 0$ $(z'(t) \leq 0)$ *for* $t \in [T, t_\ell] \cup \left(\cup_{k=\ell}^{+\infty}(t_k, t_{k+1}]\right)$, *where* $\ell = \min\{k : t_k \geq T\}$.

Proof The proof of the lemma can be found in [15].

The following theorem is the main result of this paper.

Theorem 2.1. If condition (2.10), and the following condition holds,

$$\lim_{t\to+\infty}\int_{t_0}^{t}\prod_{t_0<t_k<l}\frac{a_k^*}{b_k}r(t_k)\exp\left(\int_l^t\frac{p(s)}{r(s)}ds\right)F(l)dl=+\infty, \tag{2.11}$$

where

$$F(l)=\frac{\epsilon g_0q(t)+\sum\limits_{i=1}^{n}\exp(-\delta w(t_1))g_1\ \epsilon_iq_i(t)}{r(t)},$$

then every solution of the problem (E), $(B1)$ oscillates in G.

Proof Let $u(x,t)$ be a non-oscillatory solution of (E), $(B1)$. Without loss of generality, we can assume that there exists $T>0$, $t_0\geq T$, such that $u(x,t)>0$, $u(x,\sigma_i(t))>0$, $i=1,2,\cdots,n$, $u(x,\rho_j(t))>0$, $j=1,2,\cdots,m$ for any $(x,t)\in\Omega\times[t_0,\infty)$. From Lemma (2.1), we know that $z(t)$ is a positive solution of (2.1).

For $t\geq t_0$, $t\neq t_k$, $k=1,2,...$, define

$$w(t)=r(t)\frac{z'(t)}{z(t)},\quad t\geq t_0. \tag{2.12}$$

From Lemma (2.3), we have $w(t)\geq 0,\ t\geq t_0$, $r(t)z'(t)-w(t)z(t)=0$. We may assume that $z(t_0)=1$, thus in view of (2.1) we have that for $t\geq t_0$,

$$z(t)\ =\exp\left(\int_{t_0}^{t}w(s)ds\right), \tag{2.13}$$

$$z'(t)\ =w(t)\exp\left(\int_{t_0}^{t}w(s)ds\right), \tag{2.14}$$

we substitute (2.13)-(2.14) into (2.1) and obtain,

$$\begin{aligned}r'(t)w(t)\exp\left(\int_{t_0}^{t}w(s)ds\right)+r(t)\left[w^2(t)\exp\left(\int_{t_0}^{t}w(s)ds\right)+w'(t)\exp\left(\int_{t_0}^{t}w(s)ds\right)\right]\\+p(t)w(t)\exp\left(\int_{t_0}^{t}w(s)ds\right)+g_0\epsilon q(t)\exp\left(\int_{t_0}^{t}w(s)ds\right)\\+\sum_{i=1}^{n}g_1\epsilon_iq_i(t)\exp\left(\int_{t_0}^{\sigma_i(t)}w(s)ds\right)\leq 0.\end{aligned}$$

Hence we have

$$r(t)w^2(t) + r(t)w'(t) + p(t)w(t) + g_0\epsilon q(t) + \sum_{i=1}^{n} g_1\epsilon_i q_i(t)\exp\left(-\int_{\sigma_i(t)}^{t} w(s)ds\right) \leq 0,$$

$t \neq t_k$, or

$$r(t)w'(t) + p(t)w(t) + g_0\epsilon q(t) + \sum_{i=1}^{n} g_1\epsilon_i q_i(t)\exp\left(-\int_{\sigma_i(t)}^{t} w(s)ds\right) \leq 0,\ t \neq t_k.$$

From above inequality and condition $b_k \leq a_k^*$, it is easy to see that the function $w(t)$ is non-increasing for $t \geq t_1 \geq \delta + t_0$. Thus $w(t) \leq w(t_1)$ for $t \geq t_1$ which implies that

$$r(t)w'(t) + p(t)w(t) + g_0\epsilon q(t) + \exp\left(-\delta w(t_1)\right)\sum_{i=1}^{n} g_1\epsilon_i q_i(t) \leq 0, \quad t \neq t_k.$$

From (2.1), we obtain

$$w(t_k^+) = r(t_k^+)\frac{z'(t_k^+)}{z(t_k^+)} \leq r(t_k^+)\frac{b_k z'(t_k)}{a_k^* z(t_k)} = r(t_k)\frac{b_k}{a_k^*}w(t_k),$$

and

$$\begin{aligned} r(t)w'(t) &\leq -p(t)w(t) - g_0\epsilon q(t) - \exp\left(-\delta w(t_1)\right)\sum_{i=1}^{n} g_1\epsilon_i q_i(t), \quad t \neq t_k. \\ w(t_k^+) &\leq r(t_k)\frac{b_k}{a_k^*}w(t_k), \quad k = 1, 2, ... \end{aligned}$$

Let

$$-F(l) = \frac{-g_0\epsilon q(t) - \exp\left(-\delta w(t_1)\right)\sum\limits_{i=1}^{n} g_1\epsilon_i q_i(t)}{r(t)}.$$

Then according to Lemma (2.2), we have

$$
\begin{aligned}
w(t) &\le w(t_0) \prod_{t_0<t_k<t} r(t_k)\frac{b_k}{a_k^*} \exp\left(\int_{t_0}^{t} \frac{p(s)}{r(s)}ds\right) \\
&+ \int_{t_0}^{t} \prod_{l<t_k<t} r(t_k)\frac{b_k}{a_k^*} \exp\left(\int_{l}^{t} \frac{p(s)}{r(s)}ds\right) F(l)dl \\
&= \prod_{t_0<t_k<t} \frac{b_k}{a_k^*}\Bigg[w(t_0) r(t_k) \exp\left(\int_{t_0}^{t} \frac{p(s)}{r(s)}ds\right) \\
&- \int_{t_0}^{t} \prod_{t_0<t_k<l} r(t_k)\frac{a_k^*}{b_k} \exp\left(\int_{l}^{t} \frac{p(s)}{r(s)}ds\right) F(l)dl \Bigg] < 0.
\end{aligned}
$$

Since $w(t) \ge 0$, the last inequality contradicts condition (2.11). This completes the proof.

3. Oscillation properties of the problem (E) and $(B2)$

Next we consider the problem (E) and $(B2)$. To prove our main result we need the following lemma.

Lemma 3.1. *Let $u(x,t) \in C^2(\Gamma) \cap C^1(\bar{\Gamma})$ be a positive solution of the problem $(E), (B2)$ in G. Then the function $z(t)$ satisfies the impulsive differential inequality*

$$
\left.
\begin{aligned}
&[r(t)z'(t)]' + p(t)z'(t) + \epsilon g_0 q(t) z(t) + \sum_{i=1}^{n} \epsilon_i g_1 q_i(t) z(\sigma_i(t)) \le 0, \quad t \ne t_k \\
&a_k^* \le \frac{z(t_k^+)}{z(t_k)} \le a_k \\
&b_k^* \le \frac{z'(t_k^+)}{z'(t_k)} \le b_k, \quad t = t_k, \qquad k = 1, 2, ...
\end{aligned}
\right\} \tag{3.1}
$$

where $z(t) = v(t) + c(t)v(\tau(t))$.

Proof Let $u(x,t)$ be a positive solution of the problem (E), $(B2)$ in G. Without loss of generality, we may assume that there exists a $T > 0,\ t_0 > T$ such that $u(x,t) > 0,\ u(x,\tau(t)) > 0,\ u(x,\sigma_i(t)) > 0,\ i = 1, 2, \cdots, n,\ u(x,\rho_j(t)) > 0,\ j = 1, 2, \cdots, m$ for any $(x,t) \in \Omega \times [t_0, +\infty)$.

For $t \ge t_0,\ t \ne t_k,\ k = 1, 2, \cdots$, integrating (E) with respect to x over the domain Ω yields

$$\left.\begin{aligned}&\frac{d}{dt}\left[r(t)\frac{d}{dt}\left(\int_\Omega u(x,t)dx+\int_\Omega c(t)u(x,\tau(t))dx\right)\right]\\&+p(t)\frac{d}{dt}\left(\int_\Omega u(x,t)dx+\int_\Omega c(t)u(x,\tau(t))dx\right)\\&+\int_\Omega q(x,t)f(u(x,t))dx+\sum_{i=1}^{n}\int_\Omega q_i(x,t)f_i(u(x,\sigma_i(t)))dx\\&=\int_\Omega a(t)h(u(x,t))\Delta u(x,t)dx+\sum_{j=1}^{m}\int_\Omega b_j(t)h_j(u(x,\rho_j(t)))\Delta u(x,\rho_j(t))dx\\&\quad+\int_\Omega g(x,t)dx.\end{aligned}\right\} \tag{3.2}$$

By Green's formula, and the boundary condition $(B2)$, we have

$$\begin{aligned}\int_\Omega h(u(x,t))\Delta u(x,t)dx &= \int_{\partial\Omega} h(u(x,t))\frac{\partial u(x,t)}{\partial\gamma}dS-\int_\Omega h'(u(x,t))\left|grad\ u\right|^2dx\\&=-\int_{\partial\Omega} h(u(x,t))\mu(x,t)udS-\int_\Omega h'(u(x,t))\left|grad\ u\right|^2dx\\&=-\int_\Omega h'(u(x,t))\left|grad\ u\right|^2dx\le 0,\end{aligned} \tag{3.3}$$

and for $j=1,2,\cdots,m$

$$\begin{aligned}\int_\Omega h_j(u(x,\rho_j(t)))\Delta u(x,\rho_j(t))dx &= \int_{\partial\Omega} h_j(u(x,\rho_j(t)))\frac{\partial u(x,\rho_j(t))}{\partial\gamma}dS\\&-\int_\Omega h_j'(u(x,\rho_j(t)))\left|grad\ u\right|^2dx\\&=-\int_{\partial\Omega} h_j(u(x,\rho_j(t)))\mu(x,\rho_j(t))u(x,\rho_j(t))dx\\&-\int_\Omega h_j'(u(x,\rho_j(t)))\left|grad\ u\right|^2dx\\&=-\int_\Omega h_j'(u(x,\rho_j(t)))\left|grad\ u\right|^2dx\le 0,\end{aligned} \tag{3.4}$$

where dS is the surface element on $\partial\Omega$.

The proof is similar to that of Lemma (2.1) and therefore the details are omitted.

Using the above lemma, we prove the following oscillation result.

Theorem 3.2. *If condition (2.10) and (2.11) hold. Then each solution of* (E), $(B2)$

oscillatory in G.

Proof The proof is similar to that of Theorem (2.1), and therefore the details are omitted.

4. Example

In this section, we present an example to illustrate the main result.

Example 4.1. *Consider the impulsive differential equation*

$$\left.\begin{array}{l} \dfrac{\partial}{\partial t}\left(2t\dfrac{\partial}{\partial t}\left(u(x,t)+\frac{1}{2}u(x,t-\frac{\pi}{2})\right)\right)+(-2)\dfrac{\partial}{\partial t}\left(u(x,t)+\frac{1}{2}u(x,t-\frac{\pi}{2})\right) \\ +\dfrac{3}{2}u(x,t)+\dfrac{9\pi}{2}u(x,t-\frac{5\pi}{2})=2t\Delta u(x,t)+\left(t-\frac{9\pi}{2}\right)\Delta u(x,t-\frac{9\pi}{2}) \\ +\ g(x,t), \qquad\qquad t\neq t_k, \quad k=1,2,3,.... \\ \qquad u(x,t_k^+) = \frac{k+1}{k}u(x,t_k) \\ \qquad u_t(x,t_k^+) = u_t(x,t_k), \qquad k=1,2,.... \end{array}\right\} \quad (4.1)$$

for $(x,t)\in(0,\pi)\times[0,+\infty)$, with the boundary condition

$$u(0,t)=u(\pi,t)=0, \qquad t\neq t_k, \quad k=1,2,.... \quad (4.2)$$

Here $\Omega=(0,\pi)$, $a_k=a_k^*=\frac{k+1}{k}$, $b_k=b_k^*=1$, $k=1,2,....$ $r(t)=2t$, $c(t)=\frac{1}{2}$, $\tau(t)=t-\frac{\pi}{2}$, $p(t)=-2$, $q(t)=\frac{3}{2}$, $q_1(t)=\frac{9\pi}{2}$, $f(u)=u$, $f_1(u)=u$, $\epsilon=1$, $\sigma_1(t)=t-\frac{5\pi}{2}$, $i=1$, $a(t)=2$, $b_1(t)=1$, $h(u)=t$, $h_1(u)=t-\frac{9\pi}{2}$, $j=1$, $\rho_1(t)=t-\frac{9\pi}{2}$, $g(x,t)=\frac{3}{2}\sin x\cos t$, and taking $t_0=1$, $t_k=2^k$, $\delta=\frac{9\pi}{2}$, $w(t_1)=\frac{2}{9\pi}$. Also $g_0=\frac{1}{2}$, $g_1=\frac{1}{2}$, $F(l)=\dfrac{1}{2l}\left(\dfrac{3}{2}+\dfrac{9\pi}{4e}\right)$, we see from the above assumption that the $(H1)-(H4)$ hold, moreover

$$\begin{aligned}
\lim_{t\to+\infty}\int_{t_0}^{t}\prod_{t_0<t_k<s}\frac{b_k^*}{a_k}ds &= \int_1^{+\infty}\prod_{1<t_k<s}\frac{k}{k+1}ds \\
&= \int_1^{t_1}\prod_{1<t_k<s}\frac{k}{k+1}ds+\int_{t_1^+}^{t_2}\prod_{1<t_k<s}\frac{k}{k+1}ds \\
&\quad + \int_{t_2^+}^{t_3}\prod_{1<t_k<s}\frac{k}{k+1}ds+\int_{t_3^+}^{t_4}\prod_{1<t_k<s}\frac{k}{k+1}ds+... \\
&= 1+\frac{1}{2}\times 2+\frac{1}{2}\times\frac{2}{3}\times 2^2+\frac{1}{2}\times\frac{2}{3}\times\frac{3}{4}\times 2^3+.... \\
&= \sum_{n=0}^{+\infty}\frac{2^n}{n+1} \;=\; +\infty
\end{aligned}$$

so (2.10) holds. Thus

$$\lim_{t\to+\infty}\int_1^t \prod_{1<t_k<l}\frac{k+1}{k}(2t^k)\exp\left(-\int_l^t \frac{1}{s}ds\right)\left\{\frac{1}{2l}\left(\frac{3}{2}+\frac{9\pi}{4e}\right)\right\}dl=+\infty.$$

Hence (2.11) holds. Therefore all conditions of Theorem (2.1) are satisfied. Hence every solution of the problem (4.1), (4.2) oscillates in $(0,\pi)\times[0,+\infty)$. *In fact* $u(x,t)=\sin x\cos t$ *is one such solution of the problem (4.1) and (4.2).*

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