*South East Asian J. of Math. & Math. Sci.* **ISSN : 0972-7752**
*Vol.11, No.1 2015, pp. 39-46*

# On WP Bailey pair and transformation formulae for q-hypergeometric series

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**Abstract:** In this paper, we have established certain transformation formulae for q-hypergeometric series.
**Keywords and phrases:** WP Bailey pair, Bailey pair, summation formula, transformation formula.
**2000 A.M.S. subject classification:** . 33D10, 11F27.

## 1. Introduction, Notations and Definitions

Let q be a fixed complex parameter with $|q| < 1$. For any complex parameter 'a', the q-shifted factorial is defined by,

$$(a, q)_n = \begin{cases} 1 & \text{if } n = 0; \\ (1-a)(1-aq)(1-aq^2)\ldots,(1-aq^{n-1}) & \text{if } n \geq 1. \end{cases}$$

$$[a; q]_\infty = \prod_{r=0}^{\infty}(1 - aq^r).$$

For brevity, we write

$$[a_1, a_2, a_3, ..., a_r; q]_n = [a_1; q]_n[a_2; q]_n...[a_r; q]_n.$$

A basic (q-) hypergeometric series is defined by,

$${}_r\Phi_s\left[\begin{array}{l} a_1, a_2, ..., a_r; q; z \\ b_1, b_2, ..., b_s \end{array}\right] = \sum_{n=0}^{\infty} \frac{(a_1, a_2, ..., a_r; q)_n z^n}{(q, b_1, b_2, ..., b_s; q)_n} \left((-)^n q^{n(n-1)/2}\right)^{1+s-r}, \quad (1.1)$$

where for convergence $|z| < 1$ when $r = 1 + s$ and for $1 + s > r$, $|z| < \infty$.

Following results are needed in our analysis.

**(i)** If $A(n,r)$ is an arbitrary sequence involving integers n and r then,

$$\sum_{n=0}^{\infty}\sum_{r=0}^{n} A(n,r) = \sum_{n=0}^{\infty}\sum_{r=0}^{\infty} A(n+r,r). \tag{1.2}$$

[Srivastava & Karlsson 4; lemma 1(2) p. 100]

**(ii)** A truncated basic hypergeometric is represented as,

$$ {}_r\Phi_s\left[\begin{matrix} a_1, a_2, ..., a_r; q; z \\ b_1, b_2, ..., b_s \end{matrix}\right]_m = \sum_{n=0}^{m} \frac{(a_1, a_2, ..., a_r; q)_n z^n}{(q, b_1, b_2, ..., b_s; q)_n} \tag{1.3}$$

**(iii)**

$$ {}_6\Phi_5\left[\begin{matrix} a, q\sqrt{a}, -q\sqrt{a}, a^2/k, kq^n, q^{-n}; q; q/a \\ \sqrt{a}, -\sqrt{a}, kq/a, aq^{1-n}/k, aq^{1+n} \end{matrix}\right] = \frac{(a, aq; q)_n}{(k/a, kq/a; q)_n}\left(\frac{k}{a^2}\right)^n, \tag{1.4}$$

which can be deduced from [Gasper & Rahman 2; App. II (II.21)] by putting $c = kq^n$ and $b = a^2/k$ in it.

**(iv)** Again, taking $c = aq^{1+n}$ in [Gapser & Rahman 2; App. II. (II.21)] we get,

$$ {}_4\Phi_3\left[\begin{matrix} a, q\sqrt{a}, -q\sqrt{a}, b; q; 1/b \\ \sqrt{a}, -\sqrt{a}, aq/b \end{matrix}\right]_n = \frac{(aq, bq; q)_n}{(q, aq/b; q)_n b^n}. \tag{1.5}$$

**(v)** Taking $b = a/k$ and $c = kq^n$ in [Gasper & Rahman 2; App. II. (II.21)] we have,

$$ {}_6\Phi_5\left[\begin{matrix} a, q\sqrt{a}, -q\sqrt{a}, a/k, kq^n, q^{-n}; q; q \\ \sqrt{a}, -\sqrt{a}, kq, aq^{1-n}/k, aq^{1+n} \end{matrix}\right] = \begin{cases} 1, & \text{if } n = 0 \\ 0, & \text{if } n > 0. \end{cases} \tag{1.6}$$

**(vi)**

$$ {}_8\Phi_7\left[\begin{matrix} a, q\sqrt{a}, -q\sqrt{a}, b, c, a^2q/bck, kq^n, q^{-n}; q; q \\ \sqrt{a}, -\sqrt{a}, aq/b, aq/c, bck/a, aq^{1-n}/k, aq^{1+n} \end{matrix}\right] = \frac{(aq, aq/bc, bk/a, ck/a; q)_n}{(aq/b, aq/c, bck/a, k/a; q)_n}, \tag{1.7}$$

which can be deduced from [Gasper & Rahman 2; App. II (II.22)] by putting $d = a^2q/bck$ and $e = kq^n$.

Again, putting $k = aq$ in (1.7) we get

$$ {}_6\Phi_5\left[\begin{matrix} a, q\sqrt{a}, -q\sqrt{a}, b, c, a/bc; q; q \\ \sqrt{a}, -\sqrt{a}, aq/b, aq/c, bcq \end{matrix}\right]_n$$

$$= \frac{(aq, bq, cq, aq/bc; q)_n}{(q, aq/b, aq/c, bcq; q)_n}. \tag{1.8}$$

$${}_2\Phi_1 \left[ \begin{matrix} a, b; q; c/ab \\ cq \end{matrix} \right] = \frac{(cq/a, cq/b; q)_\infty}{(cq, cq/ab; q)_\infty} \left\{ \frac{ab(1+c) - c(a+b)}{ab - c} \right\}. \tag{1.9}$$

[Verma 5; (1.4) p. 771]

If a pair of sequences $\langle \alpha_n(a,k,q), \beta_n(a,k,q) \rangle$ satisfy

$$\alpha_0(a,k,q) = \beta_0(a,k,q) = 1$$

and

$$\beta_n(a,k,q) = \sum_{r=0}^{n} \frac{(k/a;q)_{n-r}(k;q)_{n+r}}{(q;q)_{n-r}(aq;q)_{n+r}} \alpha_r(a,k,q)$$

$$= \frac{(k, k/a; q)_n}{(q, aq; q)_n} \sum_{r=0}^{n} \frac{(q^{-n}, kq^n; q)_r}{(aq^{1+n}, aq^{1-n}/k; q)_r} \left(\frac{aq}{k}\right)^r \alpha_r(a,k,q), \tag{1.10}$$

[Laughlin 3; (1.1)]

Then these sequences are called a WP Bailey pair.

Multiplying both sides of (1.9) by $\Omega_n z^n$, summing over n from 0 to $\infty$ and applying the identity (1.2) on the right hand side we get,

$$\sum_{n=0}^{\infty} \beta_n(a,k,q)\Omega_n z^n = \sum_{n=0}^{\infty}\sum_{r=0}^{\infty} \frac{(k/a;q)_n(k;q)_{n+2r}}{(q;q)_n(aq;q)_{n+2r}} \alpha_r(a,k,q)\Omega_{n+r} z^{n+r} \tag{1.11}$$

## 2. Main Results

In this section we shall establish certain transformation formulae by making use of (1.11).

**(i)** Taking $\Omega_n = 1$ in (1.11) we get

$$\sum_{n=0}^{\infty} \beta_n(a,k,q) z^n = \sum_{r=0}^{\infty} \frac{(k;q)_{2r}}{(aq;q)_{2r}} \alpha_r(a,k,q) z^r \; {}_2\Phi_1 \left[ \begin{matrix} k/a, kq^{2r}; q; z \\ aq^{1+2r} \end{matrix} \right]. \tag{2.1}$$

Taking $z = a^2q/k^2$ in (2.1) and summing the inner ${}_2\Phi_1$ series by using [Gasper & Rahman 2; App. II (II.8)] we get,

$$\sum_{n=0}^{\infty} \beta_n(a,k,q) \left(\frac{a^2q}{k^2}\right)^n = \frac{(aq/k, a^2q/k; q)_\infty}{(aq, a^2q/k^2; q)_\infty}$$

$$\times \sum_{r=0}^{\infty} \frac{(k, kq; q^2)_r}{(a^2q/k, a^2q^2/k; q^2)_r} \left(\frac{a^2q}{k^2}\right)^r \alpha_r(a, k, q), \tag{2.2}$$

where $\langle \alpha_n(a, k, q), \beta_n(a, k, q) \rangle$ is a WP Bailey pair and $|a^2q/k^2| < 1$.
Again, taking $z = a^2/k^2$ in (2.1) and summing the inner ${}_2\Phi_1$series by using (1.9) we get

$$\sum_{n=0}^{\infty} \beta_n(a, k, q) \left(\frac{a^2}{k^2}\right)^n = \frac{k}{(k+a)} \frac{(a^2q/k, aq/k; q)_\infty}{(aq, a^2q/k^2; q)_\infty}$$

$$\times \sum_{r=0}^{\infty} \frac{(k; q)_{2r}}{(a^2q/k; q)_{2r}} \left(\frac{a^2}{k^2}\right)^r (1 + aq^{2r}) \alpha_r(a, k, q), \tag{2.3}$$

where $|a^2/k^2| < 1$.

**(a)** Choosing $\alpha_r(a, k, q) = \frac{(a, q\sqrt{a}, -q\sqrt{a}, a^2/k; q)_r}{(q, \sqrt{a}, -\sqrt{a}, kq/a; q)_r} \left(\frac{k}{a^2}\right)^r$ in (1.10) and using (1.4) we get,

$$\beta_n(a, k, q) = \frac{(a, k, q)_n}{(q, kq/a; q)_n} \left(\frac{k}{a^2}\right)^n \tag{2.4}$$

Putting these values of $\alpha_n(a, k, q)$ and $\beta_n(a, k, q)$ in (2.2) we get,

$${}_8\Phi_7 \left[ \begin{matrix} a, q\sqrt{a}, -q\sqrt{a}, \sqrt{k}, -\sqrt{k}, \sqrt{kq}, -\sqrt{kq}, \frac{a^2}{k}; q; \frac{q}{k} \\ \sqrt{a}, -\sqrt{a}, \frac{aq}{\sqrt{k}}, -\frac{aq}{\sqrt{k}}, a\sqrt{\frac{q}{k}}, -a\sqrt{\frac{q}{k}}, \frac{kq}{a} \end{matrix} \right]$$

$$= \frac{(aq, a^2q/k^2; q)_\infty}{(aq/k, a^2q/k; q)_\infty} {}_2\Phi_1 \left[ \begin{matrix} a, k; q; q/k \\ kq/a \end{matrix} \right], \quad |q/k| < 1. \tag{2.5}$$

Again, putting the values of $\alpha_n(a, k, q)$ and $\beta_n(a, k, q)$ of (2.4) in (2.3) we get,

$${}_8\Phi_7 \left[ \begin{matrix} a, q\sqrt{a}, -q\sqrt{a}, \sqrt{k}, -\sqrt{k}, \sqrt{kq}, -\sqrt{kq}, \frac{a^2}{k}; q; \frac{1}{k} \\ \sqrt{a}, -\sqrt{a}, \frac{aq}{\sqrt{k}}, -\frac{aq}{\sqrt{k}}, a\sqrt{\frac{q}{k}}, -a\sqrt{\frac{q}{k}}, \frac{kq}{a} \end{matrix} \right]$$

$$+ a\ {}_8\Phi_7 \left[ \begin{matrix} a, q\sqrt{a}, -q\sqrt{a}, \sqrt{k}, -\sqrt{k}, \sqrt{kq}, -\sqrt{kq}, \frac{a^2}{k}; q; \frac{q^2}{k} \\ \sqrt{a}, -\sqrt{a}, \frac{aq}{\sqrt{k}}, -\frac{aq}{\sqrt{k}}, a\sqrt{\frac{q}{k}}, -a\sqrt{\frac{q}{k}}, \frac{kq}{a} \end{matrix} \right]$$

$$= \left(\frac{k+a}{k}\right)\frac{(aq, a^2q/k^2;q)_\infty}{(a^2q/k, aq/k;q)_\infty}\, {}_2\Phi_1\left[\begin{matrix} a, k; q; 1/k \\ kq/a \end{matrix}\right], \tag{2.6}$$

where $|1/k| < 1$.

**(b)** Choosing $\alpha_r(a,k,q) = \dfrac{(a, q\sqrt{a}, -q\sqrt{a}, b, c, a^2q/bck;q)_r}{(q, \sqrt{a}, -\sqrt{a}, aq/b, aq/c, bck/a;q)_r}\left(\dfrac{k}{a}\right)^r$ in (1.10) and using the summation formula(1.7) we get,

$$\beta_n(a,k,q) = \frac{(k, aq/bc, bk/a, ck/a;q)_n}{(q, aq/b, aq/c, bck/a;q)_n}. \tag{2.7}$$

Putting these values of $\alpha_n(a,k,q)$ and $\beta_n(a,k,q)$ in (2.2) we have following transformation,

$$ {}_{10}\Phi_9\left[\begin{matrix} a, q\sqrt{a}, -q\sqrt{a}, \sqrt{k}, -\sqrt{k}, \sqrt{kq}, -\sqrt{kq}, b, c, \dfrac{a^2q}{bck}; q; \dfrac{aq}{k} \\ \sqrt{a}, -\sqrt{a}, \dfrac{aq}{\sqrt{k}}, -\dfrac{aq}{\sqrt{k}}, a\sqrt{\dfrac{q}{k}}, -a\sqrt{\dfrac{q}{k}}, \dfrac{aq}{b}, \dfrac{aq}{c}, \dfrac{bck}{a} \end{matrix}\right]$$

$$= \frac{(aq, a^2q/k^2;q)_\infty}{(aq/k, a^2q/k;q)_\infty}\, {}_4\Phi_3\left[\begin{matrix} k, aq/bc, bk/a, ck/a; q; a^2q/k^2 \\ aq/b, aq/c, bck/a \end{matrix}\right], \tag{2.8}$$

where $|aq/k| < 1$.

Again, putting the values of $\alpha_n(a,k,q)$ and $\beta_n(a,k,q)$ given in (2.7) in (2.3) we obtain,

$$ {}_{10}\Phi_9\left[\begin{matrix} a, q\sqrt{a}, -q\sqrt{a}, \sqrt{k}, -\sqrt{k}, \sqrt{kq}, -\sqrt{kq}, b, c, \dfrac{a^2q}{bck}; q; \dfrac{a}{k} \\ \sqrt{a}, -\sqrt{a}, \dfrac{aq}{\sqrt{k}}, -\dfrac{aq}{\sqrt{k}}, a\sqrt{\dfrac{q}{k}}, -a\sqrt{\dfrac{q}{k}}, \dfrac{aq}{b}, \dfrac{aq}{c}, \dfrac{bck}{a} \end{matrix}\right]$$

$$+ a\ {}_{10}\Phi_9\left[\begin{matrix} a, q\sqrt{a}, -q\sqrt{a}, \sqrt{k}, -\sqrt{k}, \sqrt{kq}, -\sqrt{kq}, b, c, \dfrac{a^2q}{bck}; q; \dfrac{aq^2}{k} \\ \sqrt{a}, -\sqrt{a}, \dfrac{aq}{\sqrt{k}}, -\dfrac{aq}{\sqrt{k}}, a\sqrt{\dfrac{q}{k}}, -a\sqrt{\dfrac{q}{k}}, \dfrac{aq}{b}, \dfrac{aq}{c}, \dfrac{bck}{a} \end{matrix}\right]$$

$$= \frac{(k+a)}{k}\frac{(aq, a^2q/k^2;q)_\infty}{(aq/k, a^2q/k;q)_\infty}\, {}_4\Phi_3\left[\begin{matrix} k, aq/bc, bk/a, ck/a; q; a^2/k^2 \\ aq/b, aq/c, bck/a \end{matrix}\right], \tag{2.9}$$

where $|a/k| < 1$.

**(c)** Choosing $\alpha_r(a,k,q) = \dfrac{(a, q\sqrt{a}, -q\sqrt{a}, a/k;q)_r}{(q, \sqrt{a}, -\sqrt{a}, kq;q)_r}\left(\dfrac{k}{a}\right)^r$ in (1.10) and using (1.6) we get,

$$\beta_n(a,k,q) = \frac{(k, k/a;q)_n}{(q, aq;q)_n}\delta_{n,0}. \tag{2.10}$$

Putting these values of $\alpha_n(a,k,q)$ and $\beta_n(a,k,q)$ in (2.2) we find,

$$
{}_8\Phi_7\left[\begin{matrix} a, q\sqrt{a}, -q\sqrt{a}, \sqrt{k}, -\sqrt{k}, \sqrt{kq}, -\sqrt{kq}, \frac{a}{k}; q; \frac{aq}{k} \\ \sqrt{a}, -\sqrt{a}, \frac{aq}{\sqrt{k}}, -\frac{aq}{\sqrt{k}}, a\sqrt{\frac{q}{k}}, -a\sqrt{\frac{q}{k}}, kq \end{matrix}\right]
$$

$$
= \frac{(aq, a^2q/k^2; q)_\infty}{(aq/k, a^2q/k; q)_\infty} \qquad |aq/k| < 1. \tag{2.11}
$$

Again, putting the values of $\alpha_n(a,k,q)$ and $\beta_n(a,k,q)$ from (2.10) in (2.3) we find,

$$
{}_8\Phi_7\left[\begin{matrix} a, q\sqrt{a}, -q\sqrt{a}, \sqrt{k}, -\sqrt{k}, \sqrt{kq}, -\sqrt{kq}, \frac{a}{k}; q; \frac{a}{k} \\ \sqrt{a}, -\sqrt{a}, \frac{aq}{\sqrt{k}}, -\frac{aq}{\sqrt{k}}, a\sqrt{\frac{q}{k}}, -a\sqrt{\frac{q}{k}}, kq \end{matrix}\right]
$$

$$
+ a\ {}_8\Phi_7\left[\begin{matrix} a, q\sqrt{a}, -q\sqrt{a}, \sqrt{k}, -\sqrt{k}, \sqrt{kq}, -\sqrt{kq}, \frac{a}{k}; q; \frac{aq^2}{k} \\ \sqrt{a}, -\sqrt{a}, \frac{aq}{\sqrt{k}}, -\frac{aq}{\sqrt{k}}, a\sqrt{\frac{q}{k}}, -a\sqrt{\frac{q}{k}}, kq \end{matrix}\right]
$$

$$
= \left(\frac{k+a}{k}\right) \frac{(aq, a^2q/k^2; q)_\infty}{(a^2q/k, aq/k; q)_\infty}, \tag{2.6}
$$

where $|a/k| < 1$.

## 3. Certain special transformations

In this section we establish a theorem which shall be used to find certain transformations.

If we put $k = aq$ in (1.10) and in (2.1) we get the following theorem.

**Theorem:** If

$$
\beta_n(a,q) = \sum_{r=0}^{n} \alpha_r(a,q) \tag{3.1}
$$

then

$$
\sum_{n=0}^{\infty} \beta_n(a,q) z^n = \frac{1}{(1-z)} \sum_{r=0}^{\infty} \alpha_r(a,q) z^r, \tag{3.2}
$$

where $|z| < 1$.

**(i)** Taking $\alpha_r(a,q) = \dfrac{(a, q\sqrt{a}, -q\sqrt{a}, b; q)_r}{(q, \sqrt{a}, -\sqrt{a}, aq/b; q)_r b^r}$ in (3.1) and using (1.5) we get,

$$
\beta_n(a,k) = \frac{(aq, bq; q)_n}{(q, aq/b; q)_n b^n},
$$

Putting these values in (3.2) we get,

$$(1-z)\ {}_2\Phi_1\left[\begin{array}{l} aq, bq; q; z/b \\ aq/b \end{array}\right] = {}_4\Phi_3\left[\begin{array}{l} a, q\sqrt{a}, -q\sqrt{a}, b; q; z/b \\ \sqrt{a}, -\sqrt{a}, aq/b \end{array}\right]. \tag{3.3}$$

**(ii)** Again, taking $\alpha_r(a,q) = \dfrac{(a, q\sqrt{a}, -q\sqrt{a}, b, c, a/bc; q)_r q^r}{(q, \sqrt{a}, -\sqrt{a}, aq/b, aq/c, bcq; q)_r}$ in (3.1) and using (1.8) we get,

$$\beta_n(a,k) = \frac{(aq, bq, cq, aq/bc; q)_n}{(q, aq/b, aq/c, bcq; q)_n b^n}.$$

Putting these values in (3.2) we get,

$$(1-z)\ {}_4\Phi_3\left[\begin{array}{l} aq, bq, cq, aq/bc; q; z \\ aq/b, aq/c, bcq \end{array}\right] = {}_6\Phi_5\left[\begin{array}{l} a, q\sqrt{a}, -q\sqrt{a}, b, c, a/bc; q; zq \\ \sqrt{a}, -\sqrt{a}, aq/b, aq/c, bcq \end{array}\right]. \tag{3.4}$$

## Acknowledgement

The first author is thankful to The Department of Science and Technology, Government of India, New Delhi, for support under a major research project No. SR/ S4/ MS : 735 / 2011 dated $7^{th}$ May 2013, entitled "A study of transformation theory of q-series, modular equations, continued fractions and Ramanujan's mock-theta functions," under which this work has been done

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